Entanglement of Modular Forms

Samuele Anni

René 25

Puna'auia, August 21st 2025

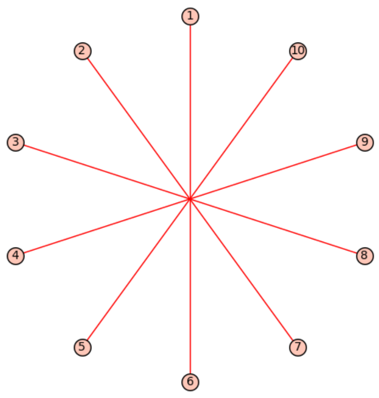
Université d'Aix-Marseille, Institut de Mathématiques

Congruences

Congruences

Definition

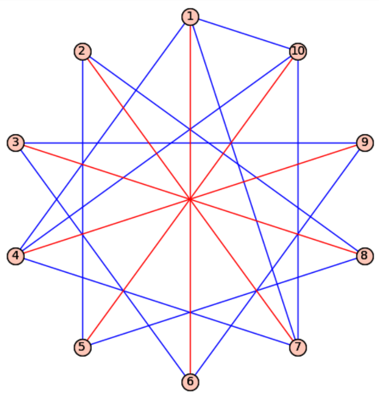
Let n be a positive integer. Two integers a and b are **congruent modulo** *n if their difference is divisible by n.*



Congruences

Definition

Let n be a positive integer. Two integers a and b are **congruent modulo** *n if their difference is divisible by n.*



Chinese remainder theorem

Theorem (Chinese remainder theorem)

Let $m_1, m_2, \ldots, m_k \in \mathbb{Z}$ be pairwise coprime positive (non-zero) integers. Then for all integers $a_1, a_2, \ldots, a_k \in \mathbb{Z}$ there exists en integer $x \in \mathbb{Z}$, unique modulo $n = \prod m_i$, such that

$$\begin{cases} x \equiv a_1 \mod m_1 \\ x \equiv a_2 \mod m_2 \\ \ldots \\ x \equiv a_k \mod m_k \end{cases} .$$

Modular forms

Modular forms

Let n be a positive integer, the congruence subgroup $\Gamma_0(n)$ is a subgroup of $\mathsf{SL}_2(\mathbb{Z})$ given by

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) : n \mid c \right\}.$$

Given a pair of positive integers n (level) and k (weight), a **modular form** f for $\Gamma_0(n)$ is an holomorphic function on the complex upper half-plane $\mathbb{H}$ satisfying

$$f(\gamma z) = f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \forall \gamma \in \Gamma_0(n), z \in \mathbb{H}$$

and a growth condition for the coefficients of its power series expansion

$$f(z) = \sum_0^\infty a_m q^m, \qquad \text{where} \qquad q = e^{2\pi i z}.$$

Newforms

There are families of operators acting on the space of modular forms. In particular, the **Hecke operators** T_p for every prime p. These operators describe the interplay between different group actions on the complex upper half-plane.

We will consider only cuspidal **newforms**: cuspidal modular forms ($a_0 = 0$), normalized ($a_1 = 1$), which are eigenforms for the Hecke operators and arise from level n.

We will denote by $S_k(n, \mathbb{C})$ the space of cuspforms and by $S_k(n, \mathbb{C})^{new}$ the subspace of newforms.

Hecke eigenvalue field

Definition

Let f be a newform, $f = \sum a_m q^m$. Then $\mathbb{Q}_f = \mathbb{Q}(\{a_m\})$ is a number field, called the **Hecke eigenvalue field** *of f.*

The set $\{a_m\}$ is a **Hecke eigenvalue system**.

Hecke eigenvalue field

Definition

Let f be a newform, $f = \sum a_m q^m$. Then $\mathbb{Q}_f = \mathbb{Q}(\{a_m\})$ is a number field, called the **Hecke eigenvalue field** *of f.*

The set $\{a_m\}$ is a **Hecke eigenvalue system**.

Example: $S_2(77, \mathbb{C})^{new}$

$f_0(q) = q - 3q^3 - 2q^4 - q^5 - q^7 + 6q^9 - q^{11} + 6q^{12} - 4q^{13} + 3q^{15} + \dots$

$f_1(q) = q + q^3 - 2q^4 + 3q^5 + q^7 - 2q^9 - q^{11} - 2q^{12} - 4q^{13} + 3q^{15} + \dots$

$f_2(q) = q + q^2 + 2q^3 - q^4 - 2q^5 + 2q^6 - q^7 - 3q^8 + q^9 - 2q^{10} + q^{11} + \dots$

$f_{3,4}(q) = q + \alpha q^2 + (-\alpha + 1)\, q^3 + 3q^4 - 2q^5 + (\alpha - 5)\, q^6 + q^7 + \dots$

where α satisfies $x^2 - 5 = 0$.

The Hecke eigenvalue fields are $\mathbb{Q}$ for f_0, f_1, f_2 and $\mathbb{Q}(\sqrt{5})$ for $f_{3,4}$.

Hecke algebra

Definition

The **Hecke algebra** $\mathbb{T}(n,k)$ *is the* $\mathbb{Z}$*-subalgebra of* $\mathrm{End}_{\mathbb{C}}(S_k(n,\mathbb{C}))$ *generated by Hecke operators* T_p *for every prime* p*.*

Newforms can be seen as ring homomorphisms $f : \mathbb{T}(n,k) \to \overline{\mathbb{Z}}$.

Congruence between newforms

Let f and g be two newforms.

$$f = \sum a_m q^m \qquad g = \sum b_m q^m.$$

Definition

We say that f and g are **congruent** *mod p, if there exists an ideal $\mathfrak{p}$ dividing p in the compositum of the Hecke eigenvalue fields of f and g such that*

$$a_m \equiv b_m \bmod \mathfrak{p} \quad \textit{for all } m.$$

Example: $S_2(77)^{new}_{\mathbb{C}}$

$$f_0(q) = q - 3q^3 - 2q^4 - q^5 - q^7 + 6q^9 - q^{11} + 6q^{12} - 4q^{13} + 3q^{15} + \dots$$
$$f_1(q) = q + q^3 - 2q^4 + 3q^5 + q^7 - 2q^9 - q^{11} - 2q^{12} - 4q^{13} + 3q^{15} + \dots$$
$$f_2(q) = q + q^2 + 2q^3 - q^4 - 2q^5 + 2q^6 - q^7 - 3q^8 + q^9 - 2q^{10} + q^{11} + \dots$$
$$f_{3,4}(q) = q + \alpha q^2 + (-\alpha + 1)\, q^3 + 3q^4 - 2q^5 + (\alpha - 5)\, q^6 + q^7 + \dots$$

where α satisfies $x^2 - 5 = 0$.

The Hecke eigenvalue fields are $\mathbb{Q}$ for f_0, f_1, f_2 and $\mathbb{Q}(\sqrt{5})$ for $f_{3,4}$.

The following congruences hold:

$f_0 \equiv f_1 \bmod 2$, $\quad f_1 \equiv f_{3,4} \bmod \mathfrak{p}_5$, $\quad f_2 \equiv f_{3,4} \bmod \mathfrak{p}_2$,

where $\mathfrak{p}_2 = (2)$, $\mathfrak{p}_5 \mid 5$ are primes in $\mathbb{Q}(\sqrt{5})$.

This is the **complete** list of possible congruences!

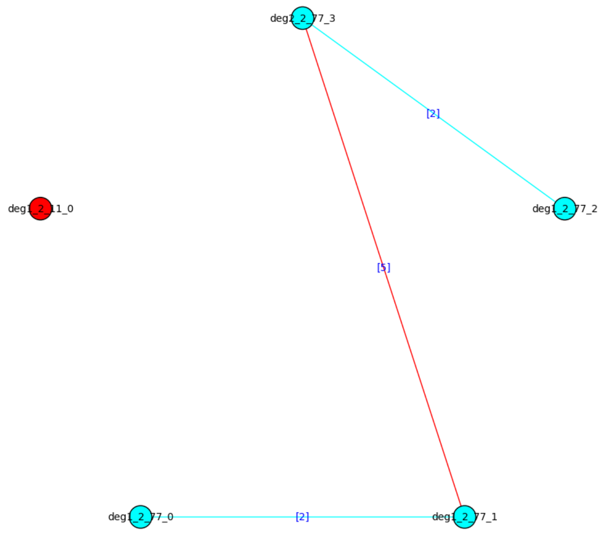
Congruence graphs

Congruence Graphs

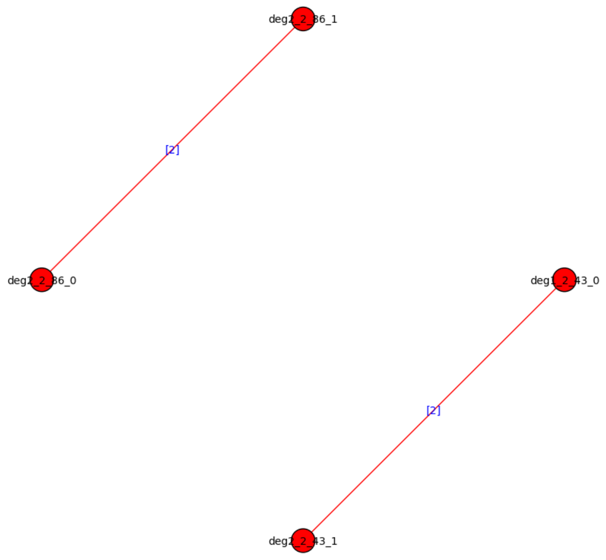
- **Nodes** correspond to Hecke orbits of newforms of level and weight in a given set (for $f \in S(n,k)^{\mathrm{new}}$ a Hecke orbit is the set of forms $\tau(f)$ for $\tau : \mathbb{Q}_f \to \overline{\mathbb{Q}}$).
- We draw an **edge** between two nodes whenever there is a prime ℓ for which there is a congruence mod ℓ between forms in the orbits.

Let S be the set of **divisors of a positive integer** and let W be a **finite set of weights**, $\mathcal{G}_{S,W}$ denotes the associated graph.

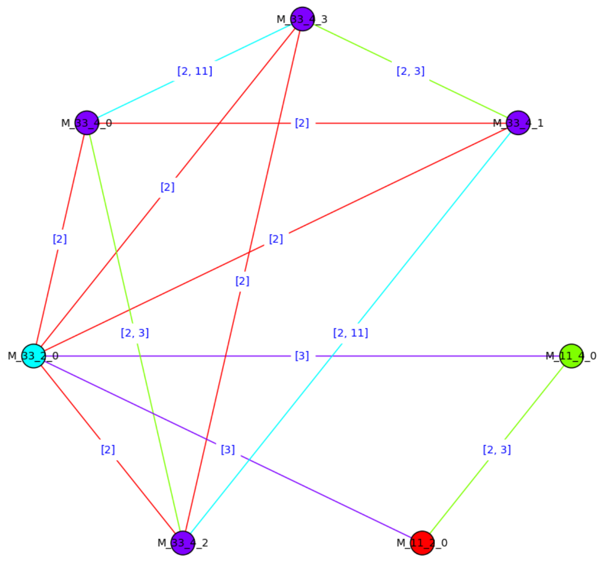
$\mathcal{G}_{[1,7,11,77],[2]}$



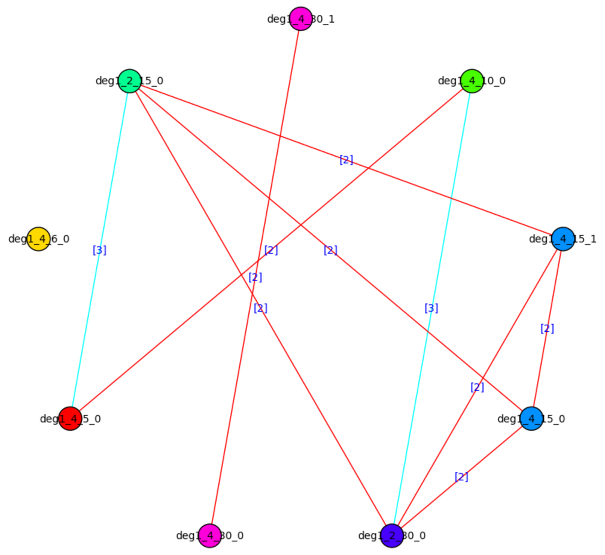
$\mathcal{G}_{[1,2,43,86],[2]}$



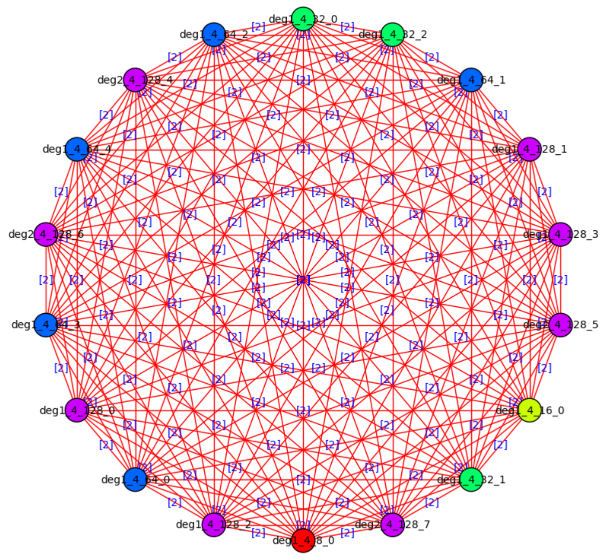
$\mathcal{G}_{[1,3,11,33],[2,4]}$



$\mathcal{G}_{[1,2,3,5,6,10,15,30],[2,4]}$



$\mathcal{G}_{[1,2,4,8,16,32,64,128],[4]}$



Conjecture

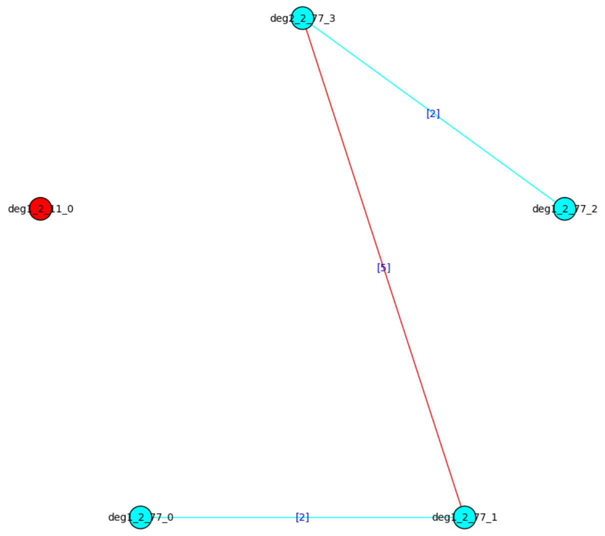
The graph $\mathcal{G}_{S,W}$ may not be connected. The computations suggest the following conjecture:

Conjecture

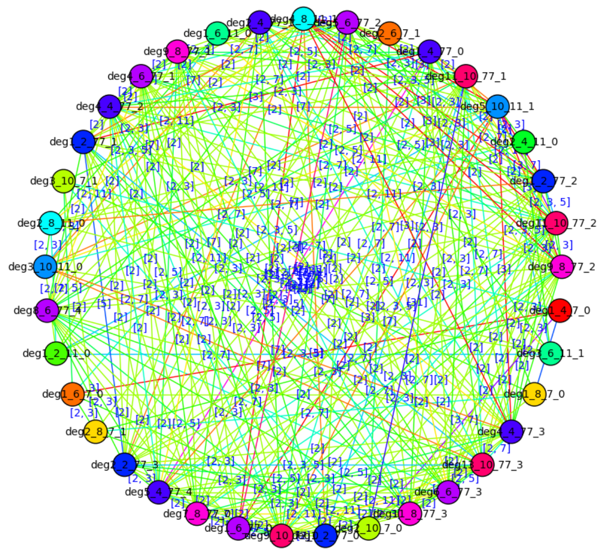
Given S and W, there exists a finite set W', with $W \subseteq W'$, such that the graph $\mathcal{G}_{S,W'}$ is connected.

This conjecture is equivalent to the connectedness of the algebra acting on the disjoint sum of newform spaces in the given set of levels and weights.

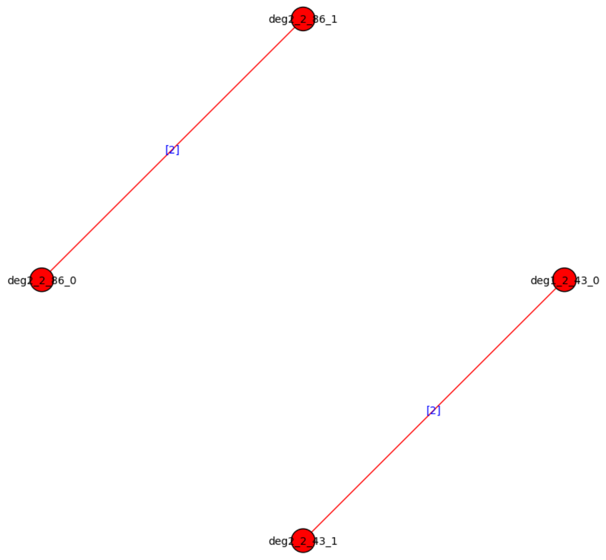
$\mathcal{G}_{[1,7,11,77],[2]}$



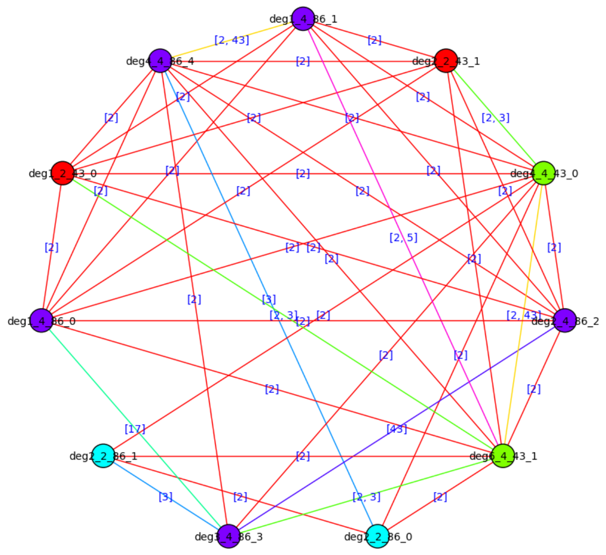
$\mathcal{G}_{[1,7,11,77],[2,4,6,8,10]}$



$\mathcal{G}_{[1,2,43,86],[2]}$



$\mathcal{G}_{[1,2,43,86],[2,4]}$



Some data

For level $n \leq 150$ we computed $\mathcal{G}_{divisors(n),2}$ and the smallest W for which $\mathcal{G}_{divisors(n),W}$ is connected. Set $k = \max(W)$.

k	Level
4	38, 45, 46, 51, 52, 54, 62, 68, 69, 72, 74, 75, 76, 86, 87, 92, 93, 94, 96, 98, 99, 100, 108, 110, 111, 116, 121, 123, 124, 134, 135, 142, 144, 147, 148, 150
6	60, 63, 90, 114, 117, 120
8	42, 55, 56, 84, 85, 95, 105, 112, 126, 140, 143
10	70, 77
12	132
14	78, 102, 104, 119, 136
20	138
2	otherwise

Residual modular Galois representations

Theorem (Deligne, Serre, Shimura)

Let n and k be positive integers. Let $\mathbb{F}$ be a finite field of characteristic ℓ, with $\ell \nmid n$, and $f : \mathbb{T}(n,k) \twoheadrightarrow \mathbb{F}$ a surjective ring homomorphism. Then there is a (unique) continuous semi-simple representation:

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}),$$

unramified outside $n\ell$, such that for all p not dividing $n\ell$ we have:

$$\mathrm{Tr}(\rho_f(\mathrm{Frob}_p)) = f(T_p) \text{ and } \det(\rho_f(\mathrm{Frob}_p)) = f(\langle p \rangle)p^{k-1} \text{ in } \mathbb{F}.$$

Such a ρ_f is unique up to isomorphism.

Field cut out by ρ_f

Given $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$ we say that K_f is the field cut out by ρ_f if the extensions $\overline{\mathbb{Q}}/K_f/\mathbb{Q}$ satisfies

$$\mathrm{Gal}(\overline{\mathbb{Q}}/K_f) = \ker(\rho_f)$$

and so ρ_f induces an isomorphism $\mathrm{Gal}(K_f/\mathbb{Q}) \cong \rho_f(G_{\mathbb{Q}}) \subseteq \mathrm{GL}_2(\mathbb{F})$.

Remarks

If f and g are congruent modulo ℓ then there exists primes $\lambda, \lambda' \mid \ell$ in $\mathbb{Q}_f$ and $\mathbb{Q}_g$, such that $\overline{\rho}_{f,\lambda} \cong \overline{\rho}_{g,\lambda'}$.

If $\overline{\rho}_{f,\lambda} \cong \overline{\rho}_{g,\lambda'}$ for $\lambda, \lambda' \mid \ell$ in $\mathbb{Q}_f$ and $\mathbb{Q}_g$, then it is not true that f and g are congruent modulo ℓ.

Example: $n_f = 38$ and $n_g = 58$

$\ell = 5$

$k_f = k_g = 2$

$n_f = 38 = 2 \cdot 19 \qquad n_g = 58 = 2 \cdot 29$

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43
$f(T_p)$	1	4	1	3	2	4	3	4	4	0	2	3	2	4
$g(T_p)$	1	4	1	3	2	4	3	0	4	4	2	3	2	4

Example: $n_f = 38$ and $n_g = 58$

$\ell = 5$

$n_f = 38 = 2 \cdot 19 \qquad n_g = 58 = 2 \cdot 29$

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43
$f(T_p)$	1	4	1	3	2	4	3	4	4	0	2	3	2	4
$g(T_p)$	1	4	1	3	2	4	3	0	4	4	2	3	2	4

It seems that $\rho_f \cong \rho_g$ since for lots of primes p we have
$\mathrm{Tr}(\rho_f(\mathrm{Frob}_p)) = f(T_p) = Tr(\rho_g(\mathrm{Frob}_p)) = g(T_p)$ and
$\det(\rho_f(\mathrm{Frob}_p)) = \epsilon_f(p) = \det(\rho_g(\mathrm{Frob}_p)) = \epsilon_g(p)$.

How can we prove this?

Computing ρ_f is "difficult", but theoretically it **can be done in polynomial time** in $n, k, \#\mathbb{F}$:

Edixhoven, Couveignes, de Jong, Merkl, Bruin, Bosman ($\#\mathbb{F} \leq 32$):

Example: for $n = 1$, $k = 22$ and $\ell = 23$, the number field corresponding to $\mathbb{P}\rho_f$ (Galois group isomorphic to $\mathrm{PGL}_2(\mathbb{F}_{23})$) is given by:

$$\begin{aligned} x^{24} &- 2x^{23} + 115x^{22} + 23x^{21} + 1909x^{20} + 22218x^{19} + 9223x^{18} + 121141x^{17} \\ &+ 1837654x^{16} - 800032x^{15} + 9856374x^{14} + 52362168x^{13} - 32040725x^{12} \\ &+ 279370098x^{11} + 1464085056x^{10} + 1129229689x^{9} + 3299556862x^{8} \\ &+ 14586202192x^{7} + 29414918270x^{6} + 45332850431x^{5} - 6437110763x^{4} \\ &- 111429920358x^{3} - 12449542097x^{2} + 93960798341x - 31890957224 \end{aligned}$$

Mascot, Zeng, Tian ($\#\mathbb{F} \leq 53$).

Example: $n_f = 38$ and $n_g = 58$

$\ell = 5$

$n_f = 38 = 2 \cdot 19 \qquad n_g = 58 = 2 \cdot 29$

Actually, one can use very few traces (14 in this case) to decide this (joint with Peter Bruin, Leiden University) and prove that

$$\rho_f \cong \rho_g \cong \rho_{E_6} \cong 1 \oplus \chi_5,$$

where χ_5 is the mod 5 cyclotomic character.

Entanglement of Modular Forms

joint work with Luis Dieulefait and Gabor Wiese

Question 1

Given $f \in S_k(n, \mathbb{C})^{new}$ and fix two primes p and q.
Let K^p and K^q be the fields cut out by $\rho_{f,p}$ and $\rho_{f,q}$ respectively.

Are K^p and K^q linearly disjoint?

Question 1

Given $f \in S_k(n, \mathbb{C})^{new}$ and fix two primes p and q.
Let K^p and K^q be the fields cut out by $\rho_{f,p}$ and $\rho_{f,q}$ respectively.

Are K^p and K^q linearly disjoint?

Question 2

Given $f \in S_{k_f}(n_f, \mathbb{C})^{new}$ and $g \in S_{k_g}(n_g, \mathbb{C})^{new}$, and two primes p and q. Let $K^{f,p}$ and $K^{g,q}$ be the fields cut out by $\rho_{f,p}$ and $\rho_{g,q}$ respectively.

Are $K^{f,p}$ and $K^{g,q}$ linearly disjoint?

Question 1

Given $f \in S_k(n, \mathbb{C})^{new}$ and fix two primes p and q.
Let K^p and K^q be the fields cut out by $\rho_{f,p}$ and $\rho_{f,q}$ respectively.

Are K^p and K^q linearly disjoint?

Question 2

Given $f \in S_{k_f}(n_f, \mathbb{C})^{new}$ and $g \in S_{k_g}(n_g, \mathbb{C})^{new}$, and two primes p and q. Let $K^{f,p}$ and $K^{g,q}$ be the fields cut out by $\rho_{f,p}$ and $\rho_{g,q}$ respectively.

Are $K^{f,p}$ and $K^{g,q}$ linearly disjoint?

If f and g are congruent modulo ℓ then certainly $K^{f,\ell} = K^{g,\ell}$!

Entanglement

We start by defining entanglement for Galois representations of dimension $n \in \mathbb{Z}_{\geq 1}$.
For $i = 1, 2$ we consider topological fields F_i and semi-simple continuous Galois representations

$$\rho_i : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(F_i)$$

as well as the fields K_i cut out by them.

Entanglement

Definition

The maximal field of entanglement for ρ_1 and ρ_2 is $K_1 \cap K_2$.

A field $E/\mathbb{Q}$ is called a **field of entanglement for** ρ_1 **and** ρ_2 if

$$E \subseteq K_1 \cap K_2.$$

We say that ρ_1 and ρ_2 are **fully entanglement** or that there is coincidence if

$$K_1 = K_2.$$

Entanglement

Definition

We say that ρ_1 and ρ_2 admit **non-abelian entanglement** if $K_1 \cap K_2$ is a non-abelian extension of $\mathbb{Q}$ (note that it is always a Galois extension).

We say that ρ_1 and ρ_2 are projectively fully entanglement if $\ker(\overline{\rho_1}) = \ker(\overline{\rho_2})$ where $\overline{\rho_i}$ is the projectivisation of ρ_i for $i = 1, 2$.

We only consider semi-simple Galois representations in dimension 2 because the application to modular forms. This means for example that we do not have to consider non-abelian subgroups of the Borel subgroup.

Examples of entanglement

Example 1

$n = 3715327360$, $k = 2$: choose f corresponding to the isogeny class of the elliptic curve over $\mathbb{Q}$ given by $E : y^2 = x^3 - x^2 - 1033x - 12438$. One can show that the maximal field of entanglement for $\rho_{f,2}$ and $\rho_{f,3}$ is $\mathbb{Q}(\sqrt{5})$.

Example 2

Modular form g with LMFDB label 5780.2.a.c : the maximal field of entanglement for $\rho_{g,2}$ and $\rho_{g,5}$ is F, with $\mathrm{Gal}(F/\mathbb{Q}) \cong S_3$.

Example 3

Examples involving weight 1 modular forms of different levels:

LMFDB label	Galois group	Entanglement: Galois group
183.1.l.a	D_{20}	D_{10}
183.1.n.a	D_{10}	
148.1.f.a	S_4	S_3
296.1.k.a	S_4	
399.1.bi.a	A_4	C_3
399.1.n.a	A_4	

Non-abelian entanglement

Theorem

Let p, q be distinct primes and consider irreducible Galois representations $\rho_p : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ and $\rho_q : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_q)$ which admit a non-abelian entanglement field. Then the following two cases can occur

(1) *The representations ρ_p and ρ_q are projectively fully entangled.*

(2) *The representations ρ_p and ρ_q are not projectively fully entangled and*

  (a) *the projective images of ρ_p and ρ_q are dihedral groups, or*

  (b) *the projective images of ρ_p and ρ_q are both S_4, or*

  (c) *one projective image is S_4 and the other one is D_n with $3 \mid n$.*

In all these cases there is a projective field of entanglement whose Galois group is an irreducible subgroup of $\mathrm{PGL}_2(\overline{\mathbb{F}}_p)$ and $\mathrm{PGL}_2(\overline{\mathbb{F}}_q)$.

Proof.

Since ρ_p and ρ_q are irreducible, their images $G_p = \rho_p(G_{\mathbb{Q}})$, $G_q = \rho_q(G_{\mathbb{Q}})$ and their projective images $\overline{G}_p$ and $\overline{G}_q$ are irreducible subgroups.

Let Q be the non-abelian Galois group of the entanglement field. Then Q is a joint quotient of G_p and G_q.

Proof.

As G_p is irreducible, its centre $Z(G_p)$ is exactly the set of scalar matrices in it, and hence it is the kernel of $G_p \twoheadrightarrow \overline{G}_p$.
The image U_p of $Z(G_p)$ under the quotient map $G_p \twoheadrightarrow Q$ is contained in $Z(Q)$, the centre of Q.

Set $\overline{Q}_p = Q/U_p$, this is a quotient of $\overline{G}_p$.
Letting $\overline{Q} = Q/Z(Q)$, we have the composition of natural surjections

$$\overline{G}_p \twoheadrightarrow \overline{Q}_p \twoheadrightarrow \overline{Q}.$$

The exact same arguments apply with q in place of p.

Proof.

We find that $\overline{Q}$ is a quotient of both $\overline{G}_p$ and $\overline{G}_q$.
$\overline{Q}$ is not cyclic, otherwise Q would be abelian.
Therefore $\overline{Q}$, $\overline{Q}_p$ and $\overline{Q}_q$ are irreducible groups.

The conclusion can now be read off from the following table:

$\overline{G}$	emb. irred. quot.	$\overline{G}^{\mathrm{ab}}$	possible entanglem.
$\mathrm{PGL}_2(\mathbb{F}_{p^r}), p^r \geq 5$ odd	-	C_2	ab.
$\mathrm{PSL}_2(\mathbb{F}_{p^r}), p^r \geq 7$	-	1	-
A_5	A_5	1	non-ab.
S_4	S_4, D_3	C_2	non-ab, ab.
A_4	A_4	C_3	non-ab, ab.
$D_n, n \geq 3$ odd	$D_m, m \geq 3, m \mid n$	C_2	non-ab, ab.
$D_n, n \geq 2$ even	$D_m, m \geq 2, m \mid n$	$C_2 \times C_2$	non-ab, ab.

Artin representations

Any Artin representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ is semi-simple and one can always find a $\overline{\mathbb{Z}}$-lattice such that, after conjugation by an element of $\mathrm{GL}_2(\mathbb{C})$, we have $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Z}})$.

If we then fix a surjective ring homomorphism $\pi_p : \overline{\mathbb{Z}} \to \overline{\mathbb{F}}_p$, we can reduce ρ by composing it with $\pi : \mathrm{GL}_2(\overline{\mathbb{Z}}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ induced by π. Up to conjugation by an element of $\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ this reduction does not depend on the choice of lattice.

Proposition

Let $\rho_p : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *be an irreducible Galois representation. Assume that its projective image* $\overline{G}$ *is isomorphic to a subgroup of* $\mathrm{PGL}_2(\mathbb{C})$*. Then* ρ_p *admits a lift to an Artin representation* $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$*. Moreover,* $\ker(\rho) = \ker(\rho_p)$*, i. e. the fields cut out by* ρ_p *and its lift* ρ *are the same.*

Klein: Any finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$ is isomorphic to one of the following groups:

- a cyclic group C_n;
- a dihedral group D_n of order $2n$, $n \geq 2$;
- the tetrahedral group A_4;
- the octahedral group S_4;
- the icosahedral group A_5.

Proposition

(a) *Any pair of isomorphic finite subgroups* $\overline{G_1}, \overline{G_2} \subset \mathrm{PGL}_2(\mathbb{C})$ *is conjugate by a matrix class in* $\mathrm{PGL}_2(\mathbb{C})$.

(b) *Let* p *be a prime. Any pair of isomorphic finite irreducible subgroups* $\overline{G_1}, \overline{G_2} \subset \mathrm{PGL}_2(\overline{\mathbb{F}}_p)$ *is conjugate by a matrix class in* $\mathrm{PGL}_2(\overline{\mathbb{F}}_p)$.

In our application to Galois representations we need to keep track of embeddings of groups.

Proposition

Let $\overline{G_1}, \overline{G_2}$ *be finite subgroups of* $\mathrm{PGL}_2(\mathbb{C})$ *and suppose there is an isomorphism* $\alpha : \overline{G_1} \to \overline{G_2}$.

Then there exists a matrix class $C \in \mathrm{PGL}_2(\mathbb{C})$ *and a field automorphism* $\sigma : \mathbb{C} \to \mathbb{C}$ *such that the restriction to* $\overline{G_1}$ *of the automorphism* $\mathrm{PGL}_2(\sigma) \circ \mathrm{conj}_C$ *equals* α, *where* conj_C *and* $\mathrm{PGL}_2(\sigma)$ *are the automorphisms of* $\mathrm{PGL}_2(\mathbb{C})$ *given by conjugation by* C *and* σ, *respectively.*

Let p, q be distinct primes and fix surjections $\pi_p : \overline{\mathbb{Z}} \twoheadrightarrow \overline{\mathbb{F}}_p$ and $\pi_q : \overline{\mathbb{Z}} \twoheadrightarrow \overline{\mathbb{F}}_q$.

Proposition

Let $\rho_p : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ and $\rho_q : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_q)$ be irreducible Galois representations and assume that they are projectively fully entangled. Then there exists an Artin representation

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Z}})$$

lifting ρ_p:

$$\rho_p = \pi_p \circ \rho,$$

as well as a Dirichlet character $\chi : G_{\mathbb{Q}} \to \overline{\mathbb{Z}}^{\times}$ and a field automorphism $\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ such that $\sigma(\rho \otimes \chi)$ lifts ρ_q

$$\rho_q = \pi_q \circ \sigma(\rho \otimes \chi).$$

Applications to modular forms

Theorem

Let $f \in S_k(M;\mathbb{C})$ *and* $g \in S_\ell(N;\mathbb{C})$ *be normalised Hecke eigenforms and assume that there are distinct prime numbers* p, q *such that* $\rho_{f,p} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *and* $\rho_{g,q} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_q)$ *are irreducible and they admit non-abelian entanglement. Then*

(1) *If* $\rho_{f,p}$ *and* $\rho_{g,q}$ *are projectively fully entangled, then there exists a weight one newform* F*, a Galois automorphism* $\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ *and a Dirichlet character* χ *such that*

$$f \equiv F \bmod p \quad \text{and} \quad \sigma(g) \otimes \chi \equiv F \bmod q$$

Theorem

Let $f \in S_k(M;\mathbb{C})$ and $g \in S_\ell(N;\mathbb{C})$ be normalised Hecke eigenforms and assume that there are distinct prime numbers p, q such that $\rho_{f,p} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ and $\rho_{g,q} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_q)$ are irreducible and they admit non-abelian entanglement. Then

(2) *If $\rho_{f,p}$ and $\rho_{g,q}$ are not projectively fully entangled, then there exist weight one newforms F, G such that $f \equiv F \bmod p$ and $g \equiv G \bmod q$.*

   (a) *the projective images of the Artin representations attached to F and G are both isomorphic to S_4, or*

   (b) *the projective images of the Artin representations attached to F and G are both isomorphic to dihedral groups, or*

   (c) *one of the projective images of the Artin representations attached to F and G is S_4, the other one a dihedral group D_n with $3 \mid n$.*

Application to elliptic curves

Theorem (Calegari)

Let $\ell \in \{2,3,5\}$. If $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathsf{GL}_2(\mathbb{F}_\ell)$ is an odd irreducible representation with cyclotomic determinant then ρ arises from the ℓ-torsion of an elliptic curve over $\mathbb{Q}$.

Proposition

Let $\rho_{E,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathsf{GL}_2(\mathbb{F}_p)$ be the Galois representation associated to the p-torsion of an elliptic curve $E/\mathbb{Q}$. Suppose that the image of $\rho_{E,p}$ is exceptional, then there exists an elliptic curve $E'/\mathbb{Q}$ such that $\rho_{E,p}$ and $\rho_{E',\ell}$ are projectively entangled, where $\ell \in \{2,3,5\}$. If $\rho_{E,p}$ and $\rho_{E',\ell}$ are projectively fully entangled, then $\ell \in \{3,5\}$.

Application to abelian varieties

Analogous statements as for elliptic curves but ....

[10] Schoof, R.: Semistable abelian varieties with good reduction outside 15, *Manuscripta Mathematica*, **139** (2012), 49–70.

[11] Schoof, R.: Abelian varieties over real quadratic fields with good reduction everywhere, in preparation.

GRH and finite flat group schemes over $\mathbb{Z}$, Dembélé & Schoof, JTNB 2024

Entanglement of Modular Forms

Samuele Anni

René 25

Puna'auia, August 21st 2025

Thanks!

Happy birthday René !

Aix-Marseille université

INSTITUT DE MATHÉMATIQUES de MARSEILLE