

On the geometry of period lattices for elliptic curves

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Happy birthday, René!



CIMPA School on "Lattices, Heights and Diophantine Approximation," Urgench, Uzbekistan

Similarity classes of planar lattices

Every $A \in \mathrm{GL}_2(\mathbb{R})$ is a basis matrix for some planar lattice

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Two lattices Ω and Γ are said to be **similar**, denoted $\Omega \sim \Gamma$, if $\Omega = \alpha U\Gamma$ for some positive real constant α and orthogonal matrix U .

Every lattice $\Omega \in \mathcal{L}_2$ is similar to a unique lattice of the form

$$\Gamma_\tau := \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mathbb{Z}^2 \text{ for some } \tau := a + bi \text{ in}$$

$$\mathcal{F} := \{\tau = a + bi \in \mathbb{C} : 0 \leq a \leq 1/2, b \geq 0, |\tau| \geq 1\}.$$

We refer to $\mathcal{F}$ as the **set of similarity classes** of lattices in $\mathcal{L}_2$.

Elliptic curves and isogenies

Given lattices $\Lambda, \Lambda' \subset \mathbb{C}$ a nonzero morphism $\mathcal{E} \rightarrow \mathcal{E}'$ between the corresponding elliptic curves $\mathcal{E} = \mathbb{C}/\Lambda$ and $\mathcal{E}' = \mathbb{C}/\Lambda'$ which takes 0 to 0 is called an **isogeny**.

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$$\deg(\beta) = [\Lambda' : \beta\Lambda].$$

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This is precisely the size of its kernel. If an isogeny $\mathcal{E} \rightarrow \mathcal{E}'$ exists, then there also exists the dual isogeny $\mathcal{E}' \rightarrow \mathcal{E}$ of the same degree such that their composition is the multiplication-by-degree map, and hence the curves are called **isogenous**: this is an equivalence relation. There may exist multiple isogenies between two elliptic curves, but since degree of an isogeny is a positive integer, we can ask for an isogeny of minimal degree.

Isomorphism classes of elliptic curves

An **isomorphism** of elliptic curves is an injective isogeny, i.e. of degree one. Each elliptic curve is isomorphic to an elliptic curve $\mathcal{E}_\tau$ with period lattice

$$\Gamma_\tau = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mathbb{Z}^2 \text{ for some } \tau := a + bi \text{ in}$$

$$\mathcal{D} := \{\tau = a + bi \in \mathbb{C} : -1/2 < a \leq 1/2, b \geq 0, |\tau| \geq 1\}.$$

Further, $\mathcal{D}' := \mathcal{D} \setminus \{e^{i\theta} : \pi/2 < \theta < 2\pi/3\}$ is precisely the **set of isomorphism classes** of elliptic curves.

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This set $\mathcal{D}$ can also be viewed as a fundamental domain for the action of the group $SL_2(\mathbb{Z})$ on the set of lattices Γ_τ by right matrix multiplication by g^{-1} for each $g \in SL_2(\mathbb{Z})$:

$$\Gamma_\tau = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mathbb{Z}^2 \mapsto g \cdot \Gamma_\tau := \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} g^{-1} \mathbb{Z}^2.$$

Arithmetic, well-rounded, semi-stable lattices

A lattice $\Gamma = A\mathbb{Z}^2$ is called **arithmetic** if the matrix $A^t A$ is a scalar multiple of an integral matrix: this property is independent of the choice of the basis matrix A .

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Successive minima of Γ are real numbers $0 < \lambda_1(\Gamma) \leq \lambda_2(\Gamma)$:

$$\lambda_i(\Gamma) := \min \{r \in \mathbb{R}_{>0} : \dim_{\mathbb{R}} \operatorname{span}_{\mathbb{R}} (\mathbb{B}(r) \cap L) \geq i\},$$

where $\mathbb{B}(r)$ is the disk of radius r centered at the origin in $\mathbb{R}^2$. Γ is called **well-rounded (WR)** if $\lambda_1(\Gamma) = \lambda_2(\Gamma)$. WR lattices are central to discrete optimization and connected areas.

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Γ is called **semi-stable** if

$$\lambda_1(L)^2 \geq \det(\Gamma) := |\det(A)|.$$

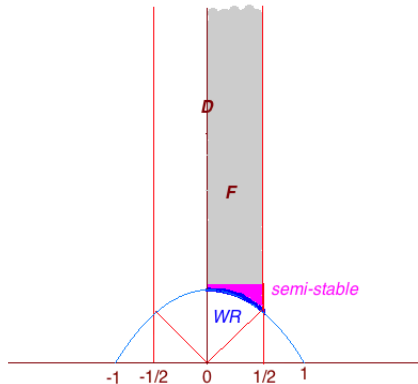
Semi-stable lattices are important in reduction theory of algebraic groups.

Geometrically speaking...

These properties of lattices are constant on similarity classes, hence we speak of arithmetic, WR, semi-stable similarity classes in $\mathcal{L}_2$, and therefore in $\mathcal{F}$.

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Algebraically speaking...

Γ_τ is arithmetic iff $\tau \in \mathcal{F}$ is of the form

$$\tau = \tau(a, b, c, d) := \frac{a}{b} + i\sqrt{\frac{c}{d}}$$

for some integers a, b, c, d such that

$$\gcd(a, b) = \gcd(c, d) = 1, \quad 0 \leq a \leq b/2, \quad c/d \geq 1 - a^2/b^2.$$

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This condition is equivalent to the elliptic curve $\mathcal{E}_\tau$ with period lattice Γ_τ having **complex multiplication** (CM) by the imaginary quadratic field $\mathbb{Q}(\tau)$, i.e. the endomorphism ring of $\mathcal{E}_\tau$ is an order in $\mathbb{Q}(\tau)$ properly containing $\mathbb{Z}$.

The j -invariant

The **Klein j -function** is a bijective holomorphic map $j : \mathcal{D}' \rightarrow \mathbb{C}$.
If $\mathcal{E}$ is an elliptic curve, then it is isomorphic to an elliptic curve $\mathcal{E}_\tau$ for precisely one $\tau \in \mathcal{D}'$, and hence the value $j(\tau)$ is an invariant of this elliptic curve, called its **j -invariant**.

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- For $\tau \in \mathcal{D}$, $j(\tau) \in \mathbb{R}$ iff τ belongs to the boundary of $\mathcal{F}$, and Γ_τ is WR iff $j(\tau) \in [0, 1]$.

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- For $\tau \in \mathcal{D}$, $j(\tau) \in \mathbb{R}$ iff τ belongs to the boundary of $\mathcal{F}$, and Γ_τ is WR iff $j(\tau) \in [0, 1]$.
- Suppose $\tau \in \mathcal{F}$ is algebraic. Then

$$\Gamma_\tau \text{ is arithmetic} \iff \deg_{\mathbb{Q}}(\tau) = 2 \iff j(\tau) \in \overline{\mathbb{Q}}.$$

In this case, the degree of the algebraic number $j(\tau)$ is the class number of the quadratic imaginary number field $\mathbb{Q}(\tau)$.

Maximum height

We define the **maximum height** of an arithmetic similarity class $\Gamma_{\tau(a,b,c,d)}$ to be

$$\mathfrak{m}(\Gamma_{\tau(a,b,c,d)}) = \mathfrak{m}(\tau(a, b, c, d)) := \max\{|a|, |b|, |c|, |d|\}.$$

Then, the number of arithmetic similarity classes with $\mathfrak{m}(\Gamma_{\tau(a,b,c,d)}) \leq T$ is finite for every real number T .

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Let $T \in \mathbb{Z}_{>0}$, and define

$$N_1(T) = |\{\Lambda_\tau : \Lambda_\tau \text{ is arithmetic and } \mathfrak{m}(\Lambda_\tau) \leq T\}|,$$

$$N_2(T) = |\{\Lambda_\tau : \Lambda_\tau \text{ is arithmetic semi-stable and } \mathfrak{m}(\Lambda_\tau) \leq T\}|,$$

$$N_3(T) = |\{\Lambda_\tau : \Lambda_\tau \text{ is arithmetic WR and } \mathfrak{m}(\Lambda_\tau) \leq T\}|.$$

Counting estimate

Theorem 1 (F., Guerzhoy, Luca (2015))

With notation as above, $N_1(T) > N_2(T) > N_3(T)$, and as $T \rightarrow \infty$,

$$N_1(T) = \frac{39T^4}{8\pi^4} + O(T^3 \log T),$$

and

$$N_2(T) = \frac{3T^4}{8\pi^4} + O(T^3 \log T),$$

while

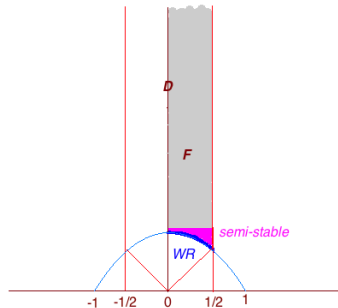
$$N_3(T) = \frac{3T^2}{2\pi^2} + O(T \log T).$$

Consequence

It follows from Theorem 1 that about 7.7% of arithmetic similarity classes in the plane are semi-stable, when counted with respect to the maximum height.

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On the other hand, only about 4.5% (with respect to Poincare measure) of all similarity classes in the plane are semi-stable.

Virtually rectangular lattices

A planar lattice M is **rectangular** if it has an orthogonal basis, i.e. if it is similar to a lattice of the form

$$M' = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}. \quad (1)$$

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L is **virtually rectangular** if it contains a rectangular sublattice M of finite index. If M is a rectangular sublattice of L of smallest index, then L is similar to L' containing a sublattice of the form M' as in (1) similar to M and

$$[L : M] = [L' : M'] = \frac{\det(L)}{|ab|}.$$

Virtually rectangular elliptic curves

Theorem 2 (F., Guerzhoy, Kühnlein (2020))

Let $\tau = a + bi \in \mathcal{D}$ and let E_τ be the corresponding elliptic curve with the period lattice Γ_τ . The following are equivalent:

1. Either $a \in \mathbb{Q}$ or there exists some $t \in \mathbb{R}$ such that $\frac{v'}{v} := a - bt, \frac{w'}{w} := a + b/t \in \mathbb{Q}$,
2. Γ_τ is virtually rectangular,
3. E_τ is isogenous to E' with real j -invariant ≥ 1 ,
4. E_τ is isogenous to E' with real j -invariant in $[0, 1]$.

If (1) - (4) hold with $a = p/q \in \mathbb{Q}$, then exists an isogeny $E' \rightarrow E_\tau$ with degree $\delta(E'/E_\tau) = q$. If (1) - (4) hold with $a \notin \mathbb{Q}$ and $t \in \mathbb{R}$ satisfies (1), then exists an isogeny $E' \rightarrow E_\tau$ with

$$\delta(E'/E_\tau) = \frac{|b||vw|(t^2 + 1)}{|t|}.$$

Remarks

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- CM elliptic curves are the only ones whose period lattice contains non-parallel rectangular sublattices: in the CM case, there are infinitely many t satisfying condition (1) of Theorem 2 (each corresponding to a different rectangular sublattice), whereas for all other virtually rectangular elliptic curves such t is essentially unique.

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- CM elliptic curves are the only ones whose period lattice contains non-parallel rectangular sublattices: in the CM case, there are infinitely many t satisfying condition (1) of Theorem 2 (each corresponding to a different rectangular sublattice), whereas for all other virtually rectangular elliptic curves such t is essentially unique.
- Virtually rectangular lattices in the plane have intrinsic geometric meaning in terms of the corresponding points on the modular curve: they correspond precisely to the points that lie on geodesics intersecting the real axis at rational points only.

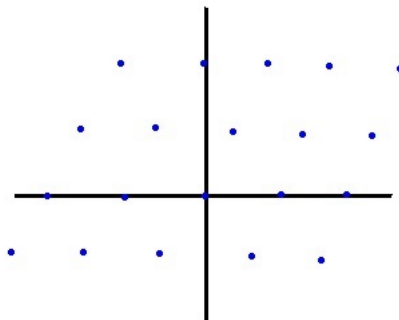
Deep holes

Let $L \subset \mathbb{R}^2$ be a lattice with successive minima $\lambda_1 \leq \lambda_2$ and the corresponding minimal basis vectors $\mathbf{x}_1, \mathbf{x}_2$. It is well known that, choosing $\pm \mathbf{x}_1, \pm \mathbf{x}_2$ if necessary, we can ensure that the angle θ between these vectors is in the interval $[\pi/3, \pi/2]$: this angle is an invariant of the lattice, we call it **angle** of L .

A **deep hole** of L is a point in $\mathbb{R}^2$ which is farthest away from the lattice. The distance from the origin to the nearest deep hole is the *covering radius* μ of L . There is a unique deep hole $\mathbf{z}$ of L contained in the triangle T with vertices $\mathbf{0}$ and the endpoints of $\mathbf{x}_1, \mathbf{x}_2$: we call it the *fundamental deep hole* of L . Define the **deep hole lattice** of L to be

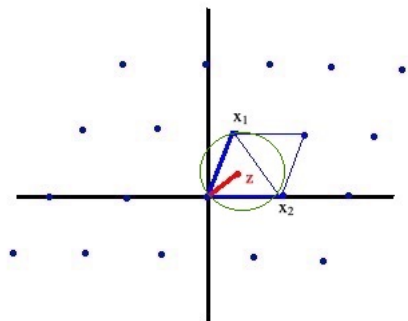
$$H(L) := \text{span}_{\mathbb{Z}}\{\mathbf{x}_1, \mathbf{z}\}.$$

Deep holes



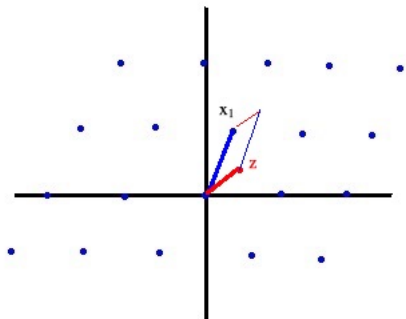
Lattice points in blue

Deep holes



Fundamental deep hole

Deep holes



Deep hole lattice: $H(L) = \text{span}_{\mathbb{Z}}\{\mathbf{x}_1, \mathbf{z}\}$.

Properties of deep hole lattices

Theorem 3 (F., Guerzhoy, Nielsen (2023))

Let L be a lattice in the plane with the angle $\theta \in [\pi/3, \pi/2]$ and successive minima λ_1 and $\lambda_2 = \alpha\lambda_1$ for some $\alpha \geq 1$. Let $H(L)$ be the deep hole lattice of L . The following statements hold:

- 1. If $\alpha \leq 2 \sin(\theta + \pi/6)$, then $H(L)$ is WR.*
- 2. If L is semi-stable, then $H(L)$ is WR.*
- 3. If L is WR, then $H(L) \sim L$.*
- 4. If $L \subset K^2$ for some subfield K of $\mathbb{R}$, then $H(L) \subset K^2$.*

Deep hole lattices in the fundamental strip

Next we turn our attention specifically to the lattices of the form Γ_τ for $\tau \in \mathcal{F}$ parameterizing all the similarity classes in the plane. Given a subfield K of $\mathbb{R}$, we say that a similarity class represented by τ **lies over** K if $\tau = a + bi$ with real numbers $a, b \in K$. This is equivalent to saying that some lattice in this similarity class is contained in $K(i) \subseteq \mathbb{C}$, which is identified with $K^2 \subseteq \mathbb{R}^2$.

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Theorem 4 (F., Guerzhoy, Nielsen (2023))

Let $\tau_0 = a_0 + b_0 i \in \mathcal{F}$ with $a_0, b_0 \in K$ for some subfield $K \subseteq \mathbb{R}$. There exists a finite sequence of numbers $\tau_1, \dots, \tau_n$ given by $\tau_k = a_k + b_k i$ for all $1 \leq k \leq n$, so that

$$a_k = \frac{1}{2}, \quad b_k = \frac{a_{k-1}^2 + b_{k-1}^2 - a_{k-1}}{2b_{k-1}} \in K \quad \forall \quad 1 \leq k \leq n, \quad (2)$$

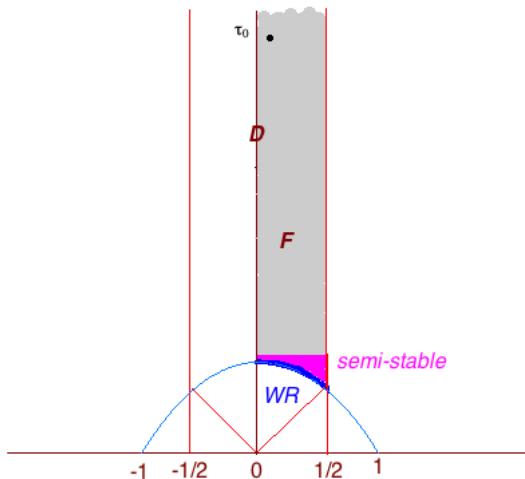
with $\Gamma_{\tau_k} = H(\Gamma_{\tau_{k-1}})$ and Γ_{τ_n} WR, hence $H(\Gamma_{\tau_n}) \sim \Gamma_{\tau_n}$. Also, $\tau_1, \dots, \tau_{n-1} \in \mathcal{F}$, $|\tau_n| \leq 1$ and $n \leq \log_2(2b_0/\sqrt{3})$.

Deep hole sequence

We call $\tau_k = a_k + b_k i$, $1 \leq k \leq n$ the **deep hole sequence** for $\tau_0 = a_0 + b_0 i \in \mathcal{F}$.

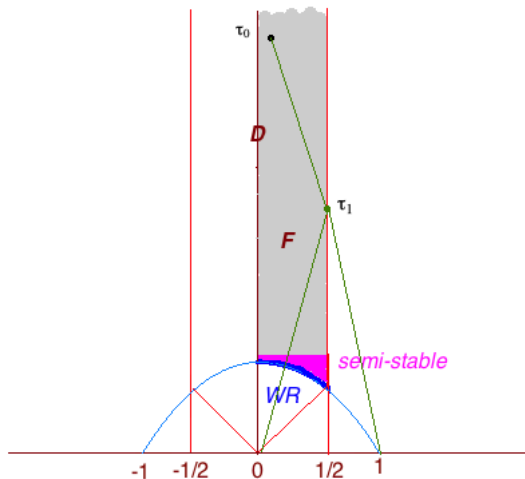
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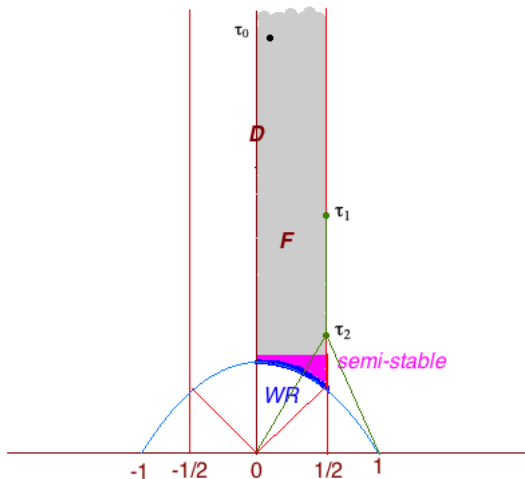
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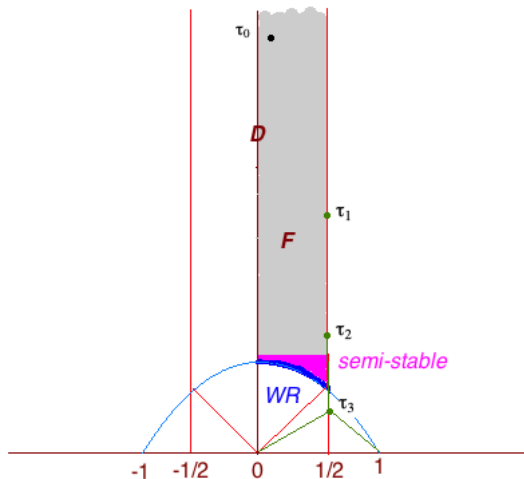
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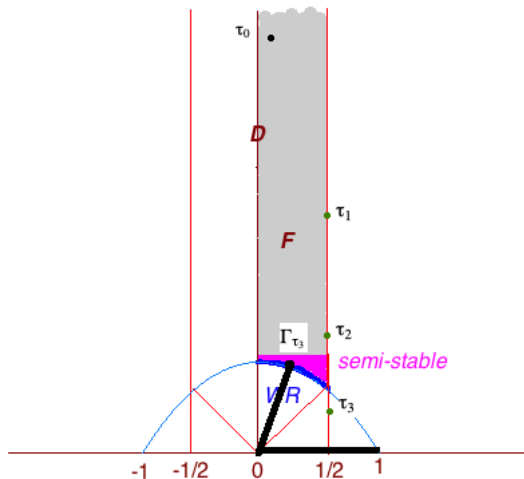
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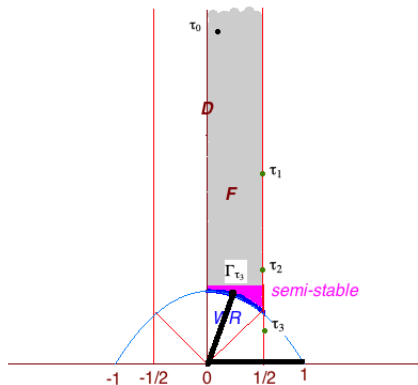


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Deep hole sequence



Thus, the map $\tau_i \rightarrow \tau_{i+1}$ defines a dynamical system, in which every point is (pre-)periodic with orbit size n as in Theorem 4.

CM case

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Theorem 5 (F., Guerzhoy, Nielsen (2023))

Let $\tau_0 = a_0 + b_0 i \in \mathcal{F}$ be a quadratic irrationality and

$$\{\tau_k = a_k + b_k i\}_{k=1}^n$$

its corresponding deep hole sequence. For each $0 \leq k \leq n$, let $\mathcal{E}_{\tau_k}$ be the corresponding CM elliptic curve with the arithmetic period lattice Γ_{τ_k} . Then all of these elliptic curves are isogenous.

Furthermore, for any $0 \leq k \leq n-1$, there exists an isogeny between $\mathcal{E}_{\tau_k}$ and $\mathcal{E}_{\tau_{k+1}}$ with degree

$$\delta_k \leq \frac{12\sqrt{3} \, b_{k+1} \, d_k^4 \, (a_k^2 + b_k^2)^2}{b_k},$$

where $d_k = \min\{d \in \mathbb{Z}_{>0} : da_k, d^2 b_k^2 \in \mathbb{Z}\}$.

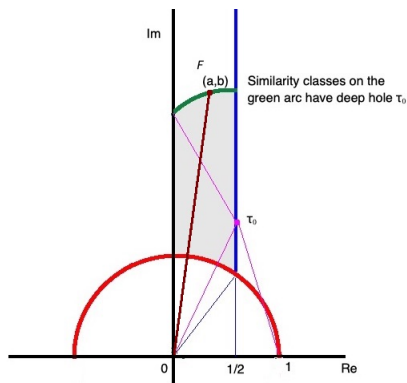
Lattices with a prescribed deep hole

Now, we consider a certain inverse problem. Let K be a number field of degree n and suppose that the similarity class represented by $\tau_0 = \frac{1}{2} + it \in \mathcal{F}$ lies over K . Consider the set

$$S_{K,\tau_0} = \{\tau \in \mathcal{F} : \tau \text{ is defined over } K \text{ and } H(\Gamma_\tau) = \Gamma_{\tau_0}\}. \quad (3)$$

i.e., the set of similarity classes defined over K whose deep hole lattice is Γ_{τ_0} . While this is an infinite set, we can count these similarity classes bounding the so-called primitive height $\mathcal{H}^P$ of τ .

Lattices with a prescribed deep hole



Similarity classes with a prescribed deep hole. Pink lines are radii of the circle centered at τ_0 . The brown line $y = \frac{b}{a}x$ intersects the green arc at a point $\tau = a + bi$ defined over K .

The primitive height

- Δ_K = be the discriminant of K
- $\mathcal{O}_K$ = ring of integers of K
- r_1 = number of real embeddings, r_2 = number of conjugate pairs of complex embeddings, so $n = r_1 + 2r_2$
- $M(K)$ = set of place of K

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For a point $\mathbf{x} \in K^m$, define its **denominator** to be

$$d(\mathbf{x}) = \min\{c \in \mathbb{Q}_{>0} : c\mathbf{x} \in \mathcal{O}_K^m\}, \quad (4)$$

and let the **(rationally) primitive point** corresponding to $\mathbf{x}$ be $\mathbf{x}_p = d(\mathbf{x})\mathbf{x}$.

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We define the **primitive height** of $\mathbf{x} \in K^m$ to be

$$\mathcal{H}^p(\mathbf{x}) := \max_{v|\infty} |\mathbf{x}_p|_v.$$

The counting estimate

Theorem 6 (F., Guerzhoy, Nielsen (2023))

For a real number $T \geq 1$, define

$$S_{K,\tau_0}(T) = \{\tau \in S_{K,\tau_0} : \mathcal{H}^P(\tau) \leq T\},$$

where $\tau_0 = \frac{1}{2} + it \in \mathcal{F}$ lies over K . Then, as $T \rightarrow \infty$,

$$|S_{K,\tau_0}(T)| \leq \left(\frac{4^{r_1} \pi^{2r_2}}{8\zeta(2n) (2t + \sqrt{4t^2 + 1}) |\Delta_K|} \right) T^{2n} + O(T^{2n-1}),$$

where ζ stands for the Riemann zeta-function and $n = [K : \mathbb{Q}]$.

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Thank you!