Challenges in generating random supersingular elliptic curves over finite fields of large characteristic

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This problem makes sense since computing the endomorphism ring of a supersingular elliptic curve over a finite field is a hard problem.

The inspiration for this talk comes from:

J. Booher, R. Bowden, J. Doliskani, T. B. Fouotsa, S. D. Galbraith, S. Kunzweiler, S.-P. Merz, C. Petit, B. Smith, K. E. Stange, Y. B. Ti, C. Vincent, J. F. Voloch, C. Weitkämper, and L. Zobernig.
Failing to hash into supersingular isogeny graphs.
https://eprint.iacr.org/2022/518.pdf, **(2022)**.
The Computer Journal, Volume 67, Issue 8, August 2024, Pages 2702–271

M. Mula, N. Murru, and F. Pintore.
On Random Sampling of Supersingular Elliptic Curves.
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Ann. Mat. Pura Appl. (4) 204, No. 3, 1293-1335 (2025).

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  We can define two operations on $\mathrm{End}(E)$. Let $\alpha, \beta \in \mathrm{End}(E)$:

$$\alpha + \beta : \begin{array}{rcl} E & \to & E \\ P & \mapsto & \alpha(P) + \beta(P) \end{array} \qquad \alpha \circ \beta : \begin{array}{rcl} E & \to & E \\ P & \mapsto & \alpha(\beta(P)) \end{array}$$

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- We denote $\mathrm{End}^0(E) := \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$. the endomorphism algebra of E.

Trivial endomorphisms

For every $n \in \mathbb{Z}$, the multiplication-by-n map

$$\begin{array}{rrcl} [n]: & E & \to & E \\ & P & \mapsto & nP = \underbrace{P + \cdots + P}_{n \text{ times}} \end{array}$$

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and $\mathrm{End}(E)$ has also the structure of a free $\mathbb{Z}$-module.

The Frobenius endomorphism

When $K = \mathbb{F}_q$ is a finite field, then we always have the Frobenius endomorphism defined as

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Attention: in some cases $\pi_E \in \mathbb{Z}$ and $\mathbb{Z}[\pi_E] = \mathbb{Z}$.

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- If E is **supersingular** then $\text{End}^0(E) \simeq B_{p,\infty}$ (the quaternion algebra ramified exactly at p and ∞) and

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where $\mathcal{O}$ is a maximal order inside $B_{p,\infty}$. In particular

$$\mathrm{End}(E) = \mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\beta + \mathbb{Z}\gamma,$$

where $\alpha, \beta, \gamma \in \mathrm{End}(E)$ are nontrivial endomorphisms.

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- Two proposals to the NIST Post-Quantum Cryptography standardization process:
  - **SIDH/SIKE** (2017): key exchange protocol, broken in 2022.
  - **SQI-Sign** (2023): signature scheme, now in round 2.

Equivalent supersingular computational problems

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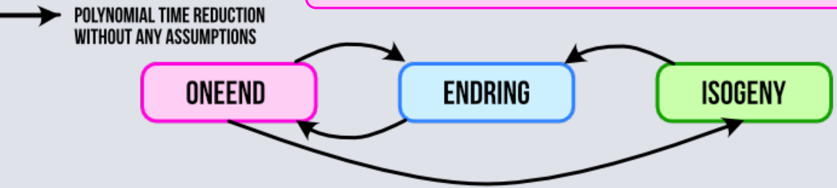

2018 - K. Eisenträger, S. Hallgren, K. Lauter, T. Morrison, and C. Petit
Supersingular isogeny graphs and endomorphism rings: Reductions and solutions.

Advances in Cryptology – EUROCRYPT 2018.

2020 - K. Eisenträger, S. Hallgren, C. Leonardi, T. Morrison, and J. Park.
Computing endomorphism rings of supersingular elliptic curves and connections to path-finding in isogeny graphs.
Proceedings of the Fourteenth Algorithmic Number Theory Symposium.

2022 - B. Wesolowski
The supersingular isogeny path and endomorphism ring problems are equivalent.
2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)

2024 - A. Page and B. Wesolowski
The Supersingular Endomorphism Ring and One Endomorphism Problems are Equivalent.
Advances in Cryptology – EUROCRYPT 2024

2025 - A. H. Le Merdy and B. Wesolowski
The supersingular endomorphism ring problem given one endomorphism.
Preprint

2025 - A. H. Le Merdy and B. Wesolowski
Unconditional foundations for supersingular isogeny-based cryptography.
Preprint

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Supersingular ℓ-isogeny graphs

Let $p > 3$ and ℓ be primes such that $p \neq \ell$.

We denote $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$ the supersingular ℓ-isogeny graph over $\overline{\mathbb{F}}_p$ with:

- **Vertices:**

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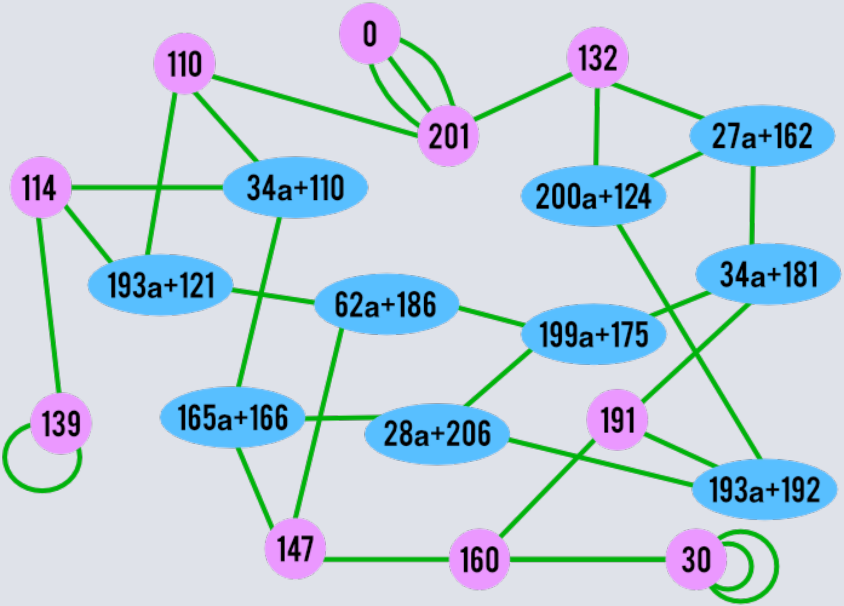
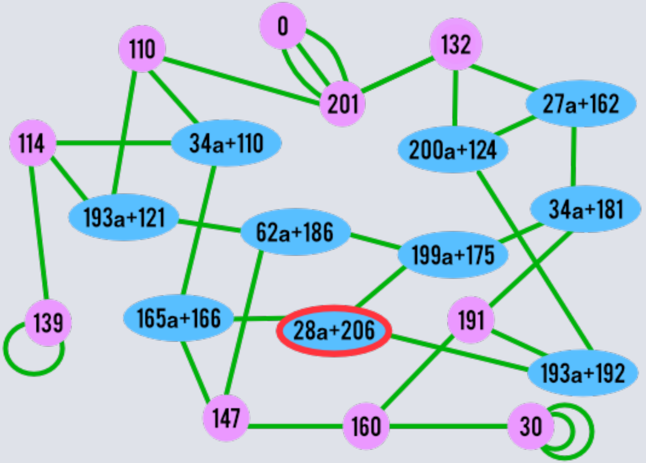
$$\text{Number of vertices} \sim \frac{p}{12}.$$

A toy example: $\mathcal{G}_2(\overline{\mathbb{F}}_{227})$



A nontrivial endomorphism in the graph



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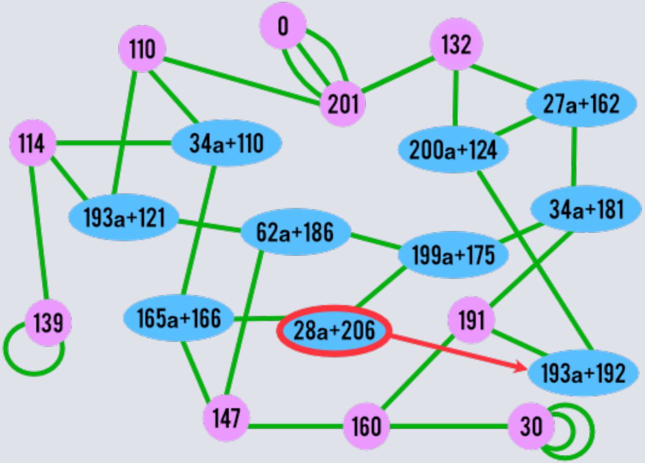

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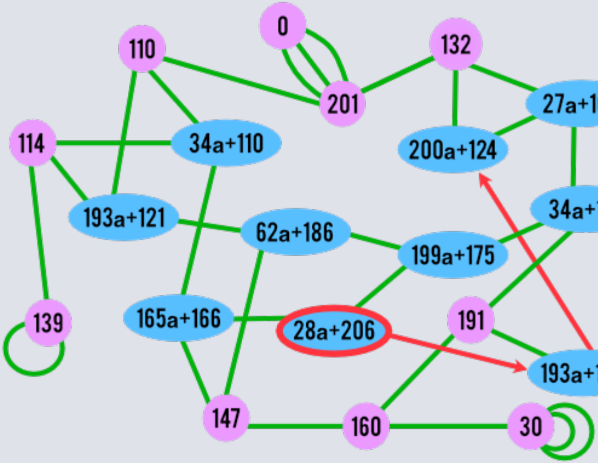

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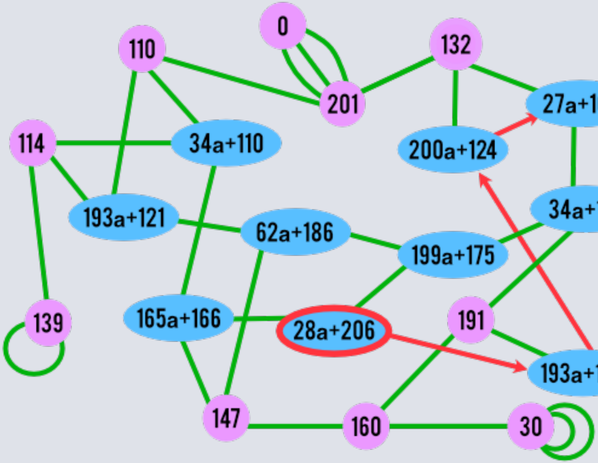

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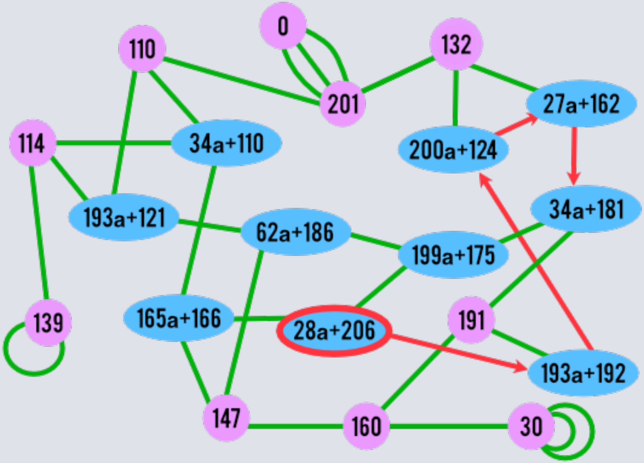

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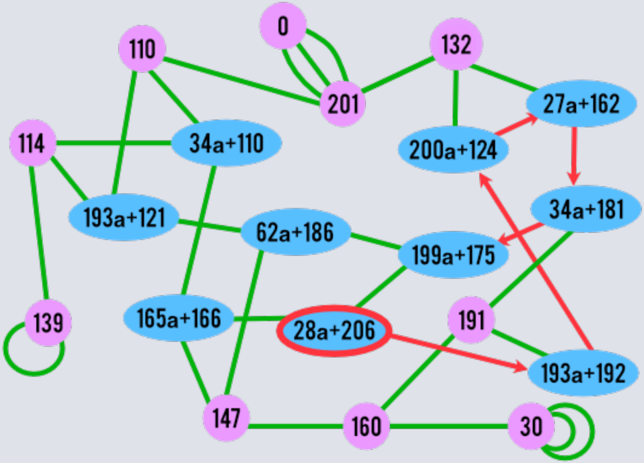

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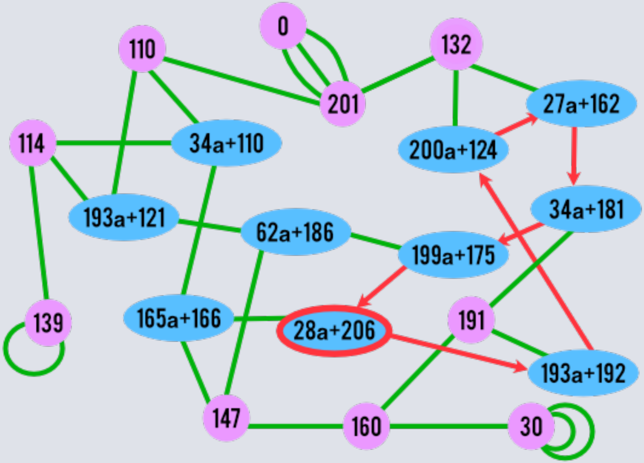

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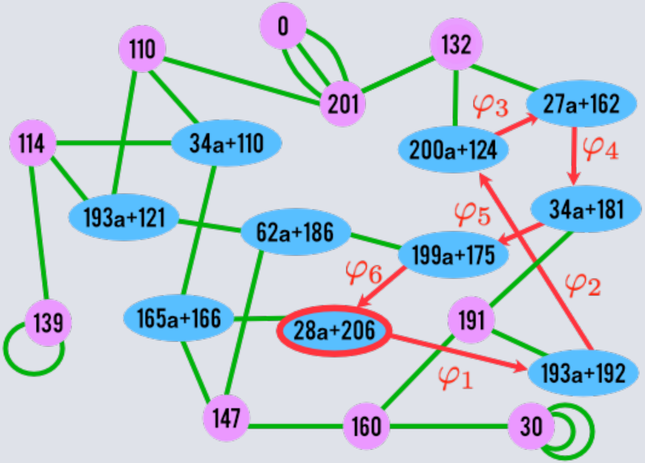

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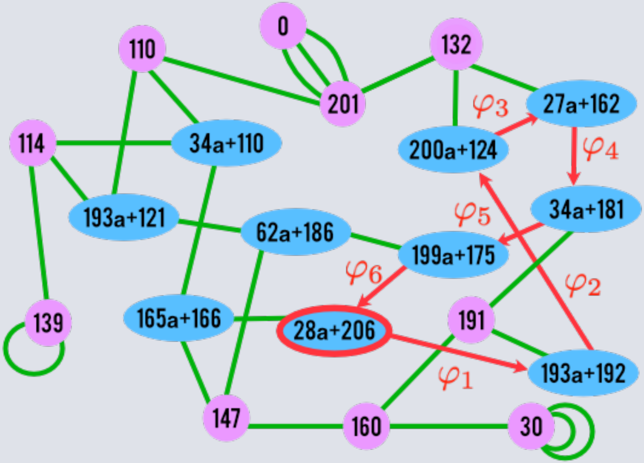


$$\varphi_6 \circ \varphi_5 \circ \cdots \circ \varphi_1 \in \mathrm{End}(E_{28a+206}).$$

Algorithms for computing nontrivial enndomorphisms

D. Kohel
Endomorphism rings of elliptic curves over finite fields.
PhD thesis, University of California, Berkeley, (1996).

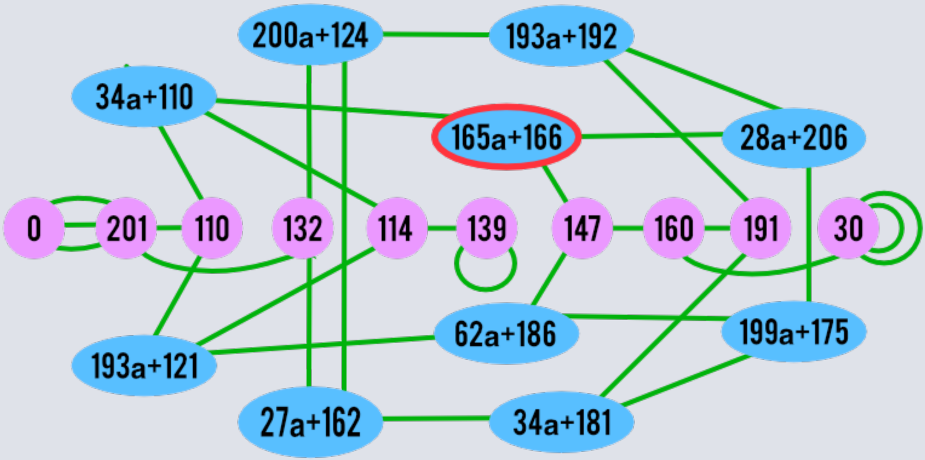
C. Delfs, and S.D. Galbraith
Computing isogenies between supersingular elliptic curves over $\mathbb{F}_p$.
Des. Codes Cryptography 78, No. 2, 425-440 (2016).

K. Eisenträger, S. Hallgren, C. Leonardi, T. Morrison, and J. Park.
Computing endomorphism rings of supersingular elliptic curves and connections to path-finding in isogeny graphs.
Proceedings of the Fourteenth Algorithmic Number Theory Symposium, pages 215–232, (2020).

A. Page and B. Wesolowski
The Supersingular Endomorphism Ring and One Endomorphism Problems are Equivalent
Advances in Cryptology – EUROCRYPT 2024

J. Fuselier, A. I., M. Kozek, T. Morrison, and C. Namoijam
Computing supersingular endomorphism rings using inseparable endomorphisms.
Journal of Algebra, Volume 668, pp 145–189, (2025)

Our idea (Fuselier, I., Kozek, Morrison, Namoijam)



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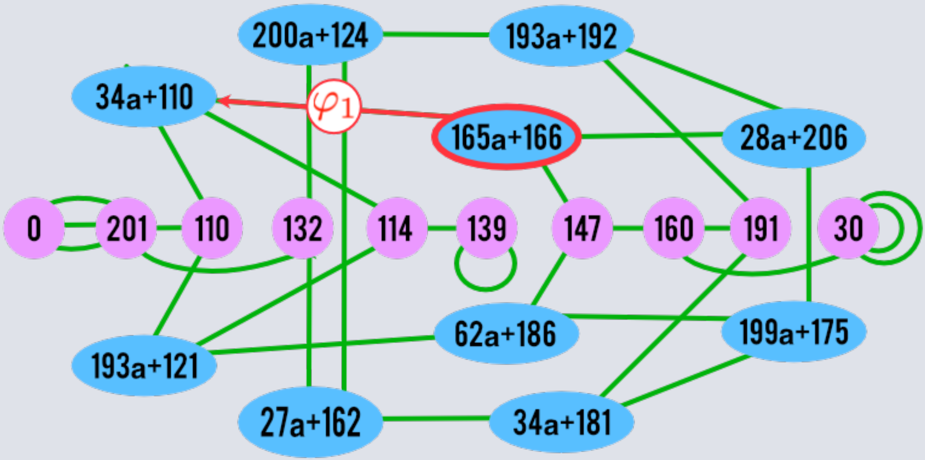

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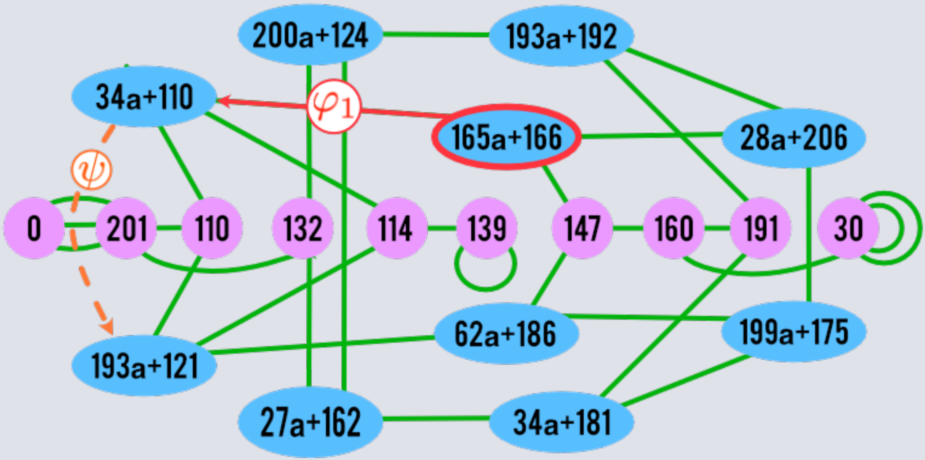

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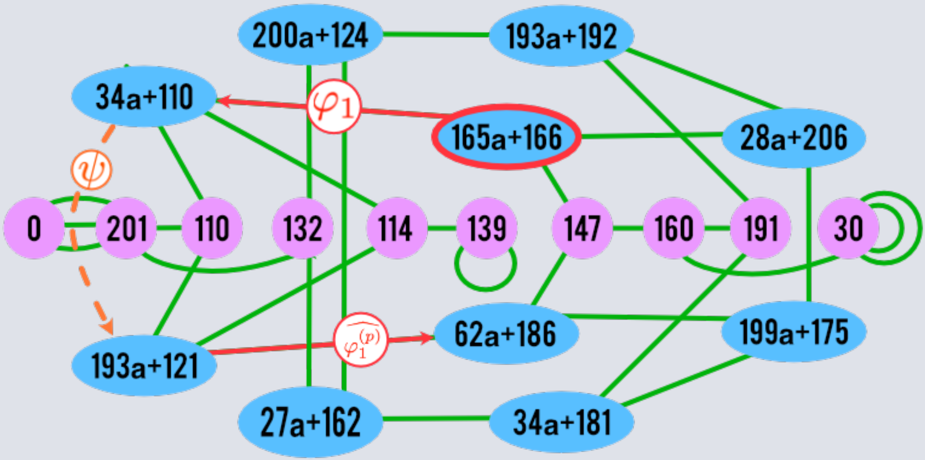

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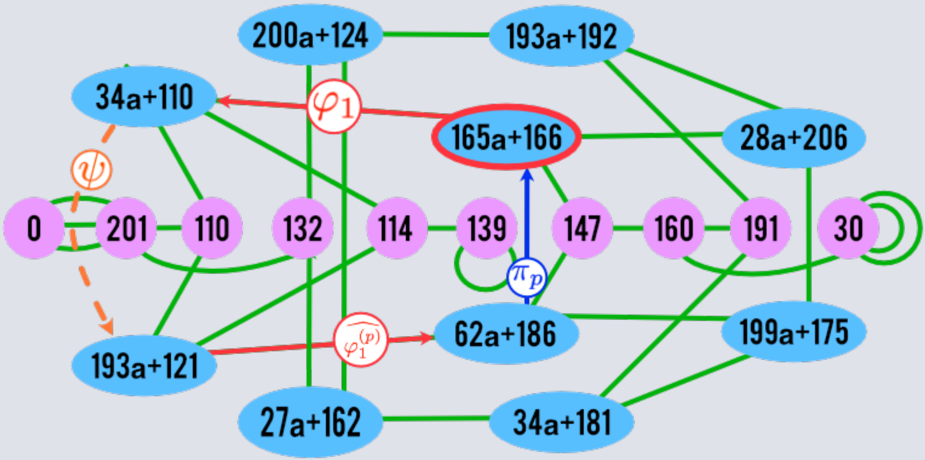

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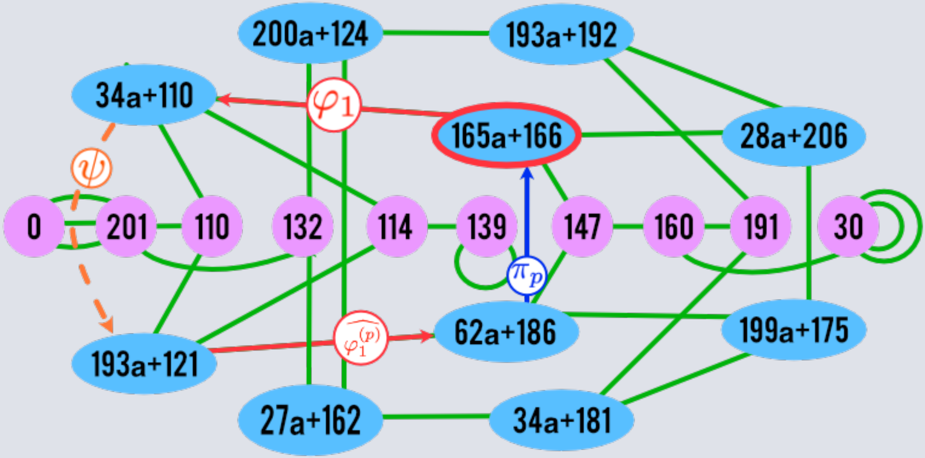
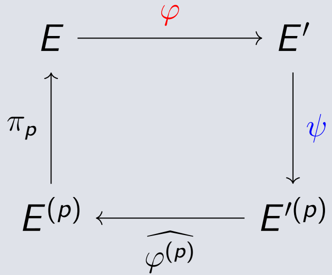


We call $\pi_p \circ \widehat{\varphi_1^{(p)}} \circ \psi \circ \varphi_1$ an *inseparable reflection*.

Inseparable reflections



- $\varphi\colon E \to E'$ is a cyclic isogeny of degree ℓ^t,
- $\psi\colon E' \to E'^{(p)}$ is an isogeny of degree d,

where ℓ is prime, and d is square-free and coprime with ℓ.

$$\alpha := \pi_p \circ \widehat{\varphi^{(p)}} \circ \psi \circ \varphi$$

is an **inseparable reflection** of degree $dp\ell^{2t}$.

Our algorithm: performance analysis

1. **Time Complexity** for computing one endomorphism:

   $$\tilde{O}(\sqrt{p}), \text{ under GRH}$$

2. **Storage requirement** for computing one endomorphism:

   $$O((\log(p))^2)$$

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3. **Towards the full endomorphism ring**: If α and β are two endormorphisms computed in this way (in two different ℓ-isogeny graphs), then:

   - α and β do not commute.
   - $\Lambda := \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z} + \alpha\beta\mathbb{Z}$ is Bass.

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Such a curve is called in the literature a hard curve.

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In any case knowing both the endomorphism rings of two supersingular elliptic curves E_1 and E_2 allows one to compute an isogeny $\varphi : E_1 \to E_2$ in polynomial time.

We'll present some ideas from:

- **J. Booher, R. Bowden, J. Doliskani, T. B. Fouotsa, S. D. Galbraith, S. Kunzweiler, S.-P. Merz, C. Petit, B. Smith, K. E. Stange, Y. B. Ti, C. Vincent, J. F. Voloch, C. Weitkämper, and L. Zobernig.**
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  Ann. Mat. Pura Appl. (4) 204, No. 3, 1293-1335 (2025).

Equivalent definitions for supersingular elliptic curves over finite fields

Theorem. Let $q = p^n$, where p is a prime number, and let E be an elliptic curve over $\mathbb{F}_q$. Then the following are equivalent:

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For p of cryptographic size, exhaustive search for supersingular j-invariants is computationally infeasible in both $\mathbb{F}_{p^2}$ and $\mathbb{F}_p$.

The cSRS problem

Given a prime p, generate uniformly random supersingular curves E over $\mathbb{F}_{p^2}$ (or equivalently superingular j-invariants in $\mathbb{F}_{p^2}$) without revealing anything about the endomorphism ring.

Crypto **S**upersingular **R**andom **S**ampling problem **(cSRS)**

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(~~c~~SRS)

Deuring's theorem (part 1)

Theorem. Let p be a prime number, $p \geq 5$.

Let E be an elliptic curve over a number field K, with $\mathrm{End}(E)$ isomorphic to an order $\mathcal{O}$ in an imaginary quadratic field L. Let $\mathfrak{p}$ be a prime ideal of K above p, and suppose that E has a good reduction modulo $\mathfrak{p}$, denoted by $\tilde{E}$.

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Rermark: p does not split over L of discriminant D if and only if $\left(\frac{D}{p}\right) \neq 1$.

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Complexity: $\tilde{O}((\log p)^3)$

Remarks

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For any p, the output belongs to a pre-determined subset of all possible supersingular j-invariants over $\mathbb{F}_{p^2}$, i.e. the roots of $H_{\mathcal{O}}$ in $\mathbb{F}_p$, which are $\tilde{O}(\sqrt{q})$.

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Random walks in the supersingular ℓ-isogeny graph over $\overline{\mathbb{F}}_p$ starting from an output of Broker's algorithm.

Deuring's theorem (part 2)

Theorem. Let $\mathcal{E}$ be an elliptic curve over $\overline{\mathbb{F}_p}$ and let $\alpha_0 \in \mathrm{End(E)} \setminus \mathbb{Z}$.

Then there exists an elliptic curve E defined over a number field K, an endomorphism α of E and a good reduction $\tilde{E}$ of E at a prime $\mathfrak{p}$ of K above p, such that $\mathcal{E}$ is isomorphic to $\tilde{E}$ and α_0 corresponds to $\tilde{\alpha}$ (the reduction of α at $\mathfrak{p}$) under the isomorphism

$$\eta : \tilde{E} \to \mathcal{E}, \qquad \eta \circ \tilde{\alpha} \circ \eta^{-1} = \alpha_0.$$

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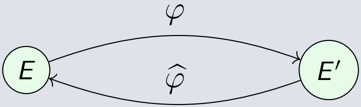

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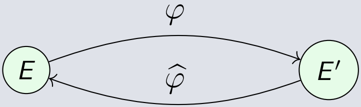

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So, the combination of Broker's algorithm and random walks solves the SRS, but not the cSRS problem.

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Theorem. Let $q = p^n$, where p is a prime number, and let E be an elliptic curve over $\mathbb{F}_q$. Then the following are equivalent:

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$$E_\lambda : y^2 = x(x-1)(x-\lambda) \qquad \text{(Legendre form)}$$

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How to find a root of $H_p(t)$ efficiently?

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How to find roots in $\mathbb{F}_{p^2}$ of the polynomial $f_{n,m,p}(x)$ efficiently?

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Compute solutions of the system in the variables x_{ℓ_i} and a:

$$\begin{cases} \psi_{\ell_i}(x_{\ell_i}, a) = 0, & \forall i = 1, \dots, r \\ x_{\ell_i}^{p^2} - x_{\ell_i} = 0, & \forall i = 1, \dots, r \end{cases}$$

where $\psi_{\ell_i}(x_i, a)$ is the division polynomial of order ℓ_i of the curve parameterized by a.

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Can techniques for constructing maximal curves be adapted or extended to produce $\mathbb{F}_{p^2}$-maximal curve of genus 1, for a large prime p?


Māuruuru roa