Structure of Supersingular Elliptic Curve Isogeny Graphs

Renate Scheidler

UNIVERSITY OF CALGARY

Joint work with **Sarah Arpin** (Virginia Tech) and **Taha Hedayat** (U Calgary)

(arXiv:2502.03613v2 [math.NT]; to appear at LuCaNT 2025 Proceedings)

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Université de la Polynèsie Française
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August 18, 2025

The Dream Team

Happy 35^{th} Birthday, René!

Motivation

Why study supersingular elliptic curve isogeny graphs? 

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Our work herein analyzes some of the structure of

- supersingular elliptic curve isogeny graphs
- their subgraphs induced by the $\mathbb{F}_p$-vertices (the *spine*)

Supersingular Isogeny Graphs

For primes $\ell \neq p$, define the ℓ-isogeny graph $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$ as follows:

- *Vertices*: $\overline{\mathbb{F}}_p$-isomorphism classes (i.e. j-invariants) of curves
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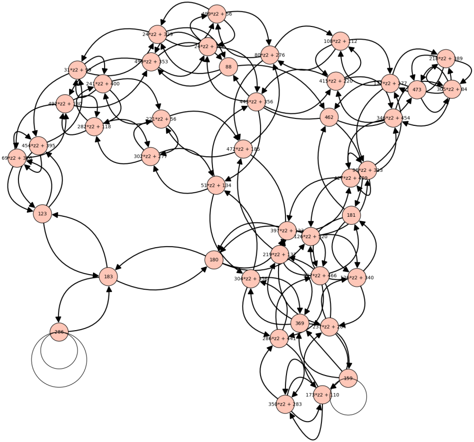
Example: $\mathcal{G}_2(\overline{\mathbb{F}}_{523})$



Connection to Cryptography

Supersingular ℓ-Isogeny Path Finding Problem

Given two supersingular elliptic curves E, E', find a path from E to E' in $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$.

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Basis for the security of the aforementioned supersingular isogeny based cryptosystems.

In practice, the path contains a sub-path of $\mathbb{F}_p$-vertices.

Motivates the study of structural properties of the spine of $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$.

Isomorphic Supersingular Elliptic Curves

Every $\overline{\mathbb{F}}_p$-isomorphism class of supersingular elliptic curves has a representative defined over $\mathbb{F}_{p^2}$

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Curves defined over $\mathbb{F}_p$ that are non-isomorphic over $\mathbb{F}_p$ can become isomorphic over $\mathbb{F}_{p^2}$:

- Example – quadratic twists: for $t^2 \in \mathbb{F}_p$, the curves

$$E : y^2 = x^3 + Ax + B \quad \text{and} \quad E_t : y^2 = x^3 + At^4x + Bt^6$$

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  are defined over $\mathbb{F}_p$ and isomorphic over $\mathbb{F}_{p^2}$ via $(x, y) \mapsto (t^2x, t^3y)$. They are isomorphic over $\mathbb{F}_p$ if and only if $t \in \mathbb{F}_p$.

Graphs from Supersingular Isogenies

For primes $\ell \neq p$, we consider three graphs:

Full supersingular ℓ-isogeny graph $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$

- *Vertices*: $\overline{\mathbb{F}}_p$-isomorphism classes (i.e. j-invariants) of supersingular elliptic curves over $\mathbb{F}_{p^2}$
- *Edges*: ℓ-isogenies* over $\overline{\mathbb{F}}_p$

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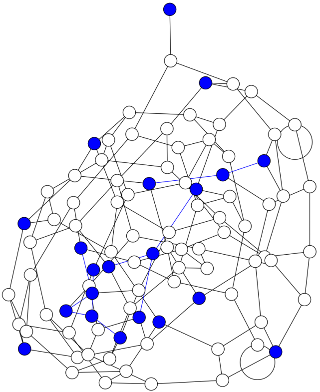
Restricted Supersingular ℓ-isogeny graph $\mathcal{G}_\ell(\mathbb{F}_p)$

- *Vertices*: $\mathbb{F}_p$-isomorphism classes (i.e. not necessarily distinct j-invariants) of supersingular elliptic curves **over** $\mathbb{F}_p$
- *Edges*: ℓ-isogenies **over** $\mathbb{F}_p$ between these vertices

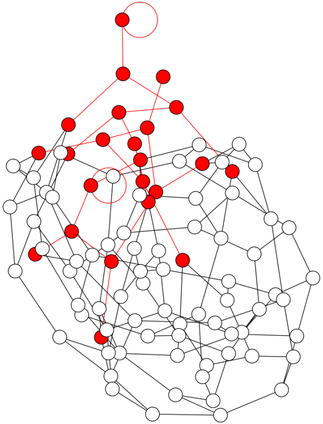
*Up to post-composition by an automorphism over $\overline{\mathbb{F}}_p$

Example: $p = 1103$, $\ell = 2$ (Courtesy Sotáková 2019)

A random graph of expected size in $\mathcal{G}_2(\overline{\mathbb{F}}_{1103})$



$\mathcal{S}_2^{1103}$

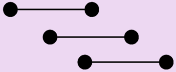
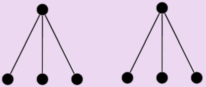
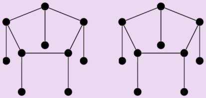


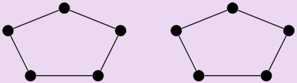
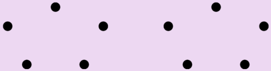
Structure of $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$ (Kohel 1996)

- Connected with approximately $p/12$ vertices
- Optimal expander graph
- Every vertex has out-degree* $\ell + 1$
- Every vertex has in-degree $\ell + 1$ except 0 and 1728 which have smaller in-degree
- By identifying isogenies with their duals, $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$ becomes an **undirected connected** graph that is $(\ell + 1)$-regular except in the neighbourhoods of vertices 0 and 1728.

*Corresponding to the $\ell + 1$ subgroups of order ℓ of the ℓ-torsion $\mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$ representing the kernels of the corresponding isogenies

Structure of $\mathcal{G}_\ell(\mathbb{F}_p)$ (Delfs & Galbraith 2013)

	$p \equiv 1 \pmod 4$	$p \equiv 3 \pmod 8$	$p \equiv 7 \pmod 8$
$\ell = 2$			

	$\left(\frac{-p}{\ell}\right) = 1$	$\left(\frac{-p}{\ell}\right) = -1$
$\ell > 2$		

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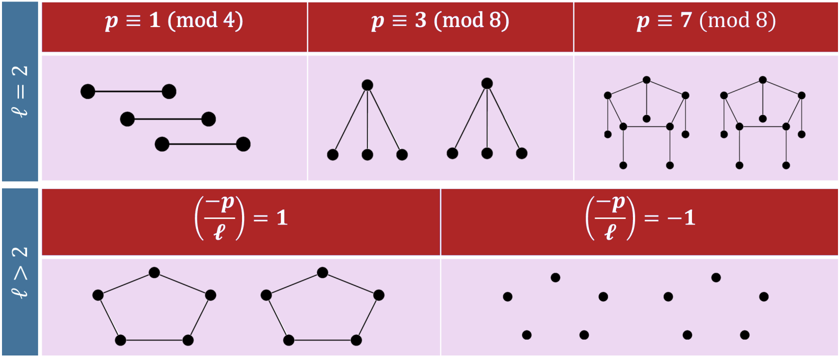

Not quite .... doesn't characterize loops or multi-edges.

Structure of $\mathcal{G}_\ell(\mathbb{F}_p)$ (H-A-S 2015)

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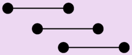
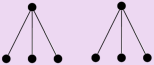
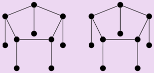
$\ell = 2$			

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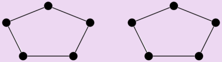
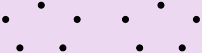
$\ell > 2$		

Loops / Multi Edges	
ℓ	p
2	7
3	5, 11
5	11, 19
7	13, 19
11	13, 19, 43
13	17, 43
17	19, 43, 59, 67
⋮	⋮

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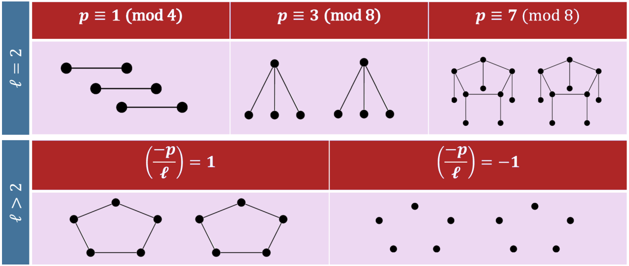

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Characterized by simple arithmetic conditions on ℓ and p.

Mapping $\mathcal{G}_\ell(\mathbb{F}_p)$ to $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$

$$\begin{array}{rcl} \mathbb{F}_p\text{-isomorphism classes} & \rightarrow & \overline{\mathbb{F}}_p\text{-isomorphism classes} \\ \mathbb{F}_p\text{-isogenies} & \rightarrow & \overline{\mathbb{F}}_p\text{-isogenies} \end{array}$$

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What happens?

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What happens?

- Pairs of vertices in $\mathcal{G}_\ell(\mathbb{F}_p)$ corresponding to quadratic twists merge into one vertex in $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$
- Isogenies defined over $\mathbb{F}_{p^2}$ but not $\mathbb{F}_p$ introduce new edges
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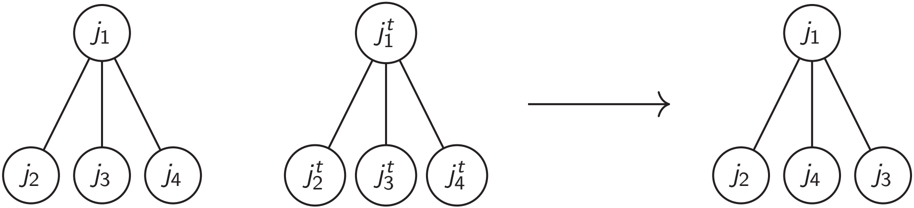
Theorem (Arpin, Camacho-Navarro, Lauter, Lim, Nelson, Scholl & Sotáková 2023)

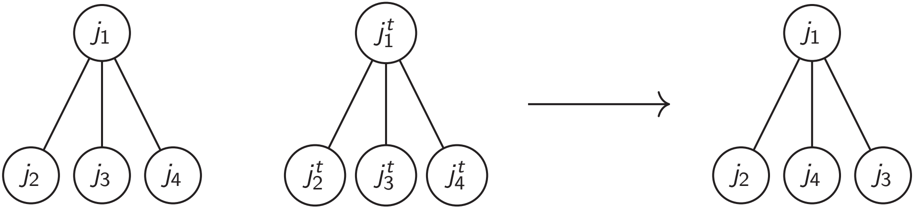
Mapping $\mathcal{G}_\ell(\mathbb{F}_p)$ to $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$ happens in 4 ways:

- Stacking
- Folding
- Attachment at a vertex
- Attachment by a new edge

Stacking



Stacking



Stacking is the default. Almost everything stacks.

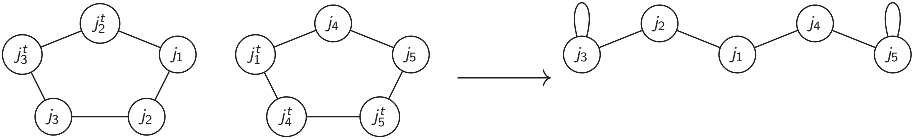
Folding

Folding

Folding and stacking are mutually exclusive.

For $\ell > 2$, only the two components of $\mathcal{G}_2(\mathbb{F}_p)$ containing 1728 fold.

Attachment at a Vertex



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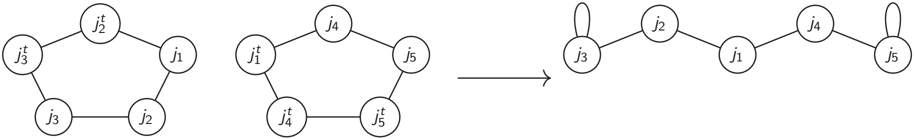
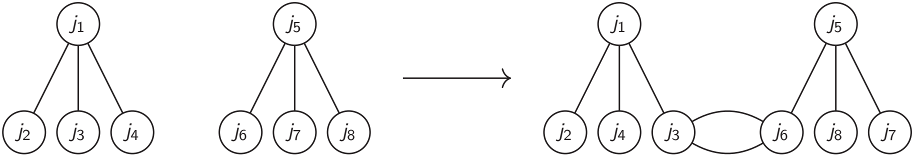


For $\ell = 2$, this does not happen.

For $\ell > 2$, the two folding components of $\mathcal{G}_2(\mathbb{F}_p)$ containing 1728 attach at 1728.

Attachment by a New Edge



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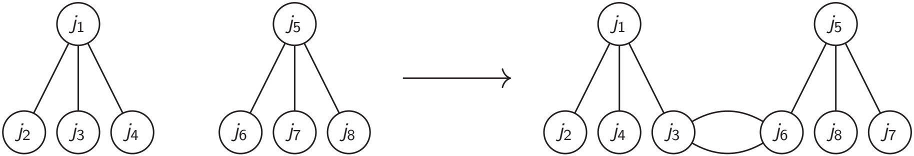

Two-for-one special: attaching edges come as double edges.

For $\ell = 2$, this happens at most once.

Mapping $\mathcal{G}_\ell(\mathbb{F}_p)$ to $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$ (H-A-S 2025)

Explicitly characterized by congruence condition on

- $p \pmod{120}$ for $\ell = 2$
- $p \pmod{840}$ for $\ell = 3$

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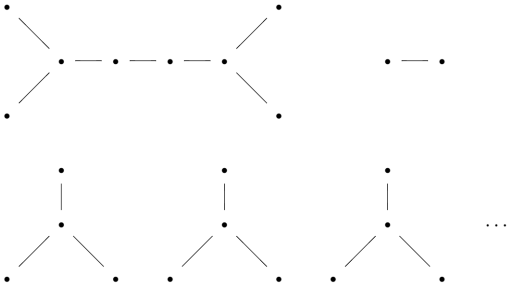
Case $\ell = 2$ and $p \equiv 11, 59 \pmod{120}$ with $p > 59$:



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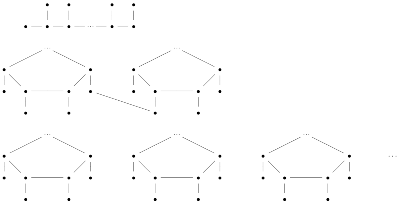

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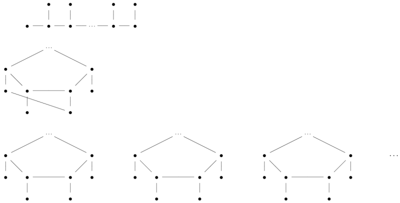

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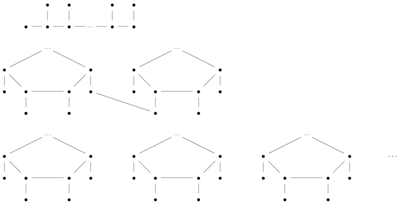

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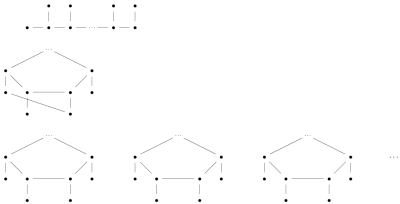

E.g. for $p = 71$ E.g. for $p = 1319$

Proof Ingredients

- ℓ-**th modular polynomial**: governs adjacency, including loops and multi-edges with multiplicities

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- **Hilbert class polynomials**: governs endomorphism ring and supersingularity
- Supersingularity of 0 and 1728
- Occasional explicit isogeny computation (to see where they are defined)
- Going nuts with Chinese Remainder Theorem

Example: $\ell = 2$

$$\begin{aligned}\Phi_2(x,y) \;=\; & -x^2y^2 + x^3 + y^3 + 1488(x^2y + xy^2) \\ & - 162000(x^2 + y^2) + 40773375xy \\ & + 8748000000(x + y) - 157464000000000\end{aligned}$$

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$$\Phi_2(x,x) = -(x + 3375)^2(x - 1728)(x - 8000)$$

Two loops at j-invariant -3375, one loop each at 1728 and 8000

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Two loops at j-invariant -3375, one loop each at 1728 and 8000

The resultant of Φ_2 and its derivative is

$$\mathrm{Res}_2(x) \;=\; -4H_{-3}(x)^2H_{-4}(x)H_{-7}(x)^2H_{-15}(x)^2 \text{ with}$$

$$H_{-3}(x) = x\;, \quad H_{-4}(x) = x - 1728\;, \quad H_{-7}(x) = x + 3375$$

$$H_{-15}(x) \;=\; x^2 + 191025x - 121287375$$

Example: $\ell = 2$

$$\begin{aligned}\Phi_2(x,y) &= -x^2y^2 + x^3 + y^3 + 1488(x^2y + xy^2)\\ &\quad - 162000(x^2+y^2) + 40773375xy\\ &\quad + 8748000000(x+y) - 157464000000000\end{aligned}$$

$$\Phi_2(x,x) = -(x+3375)^2(x-1728)(x-8000)$$

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Double edges between 0, 1728, -3375 and the roots of $H_{-15}(x)$

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- The required Hilbert class polynomials for $D = -3, -4, -8, -11, -20, -32, -35$ are still all linear or quadratic

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$\mathcal{S}_\ell^p$ for $\ell > 2$

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For $\ell > 5$: 

Diameter of $\mathcal{S}_2^p$

Diameters (lengths of longest directed path) of components of $\mathcal{S}_2^p$:

- $p \equiv 1 \pmod 4$ and $p \equiv 3 \pmod 8$: between 1 and 5
- $p \equiv 7 \pmod 8$ with $p \not\equiv 71, 119 \pmod{120}$: $(r+3)/2$ where r is order of the class of a prime $\mathbb{Z}[\sqrt{-p}]$-ideal above 2 in the class group
- $p \equiv 71, 119 \pmod{120}$: ???

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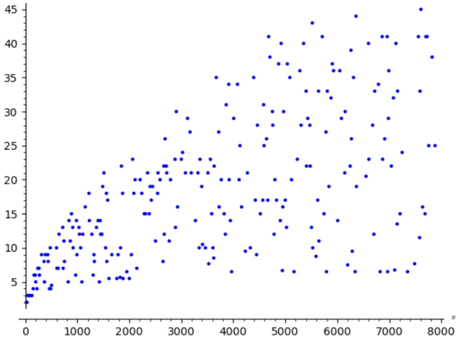

Mean component diameters in $\mathcal{S}_2^p$ for the first 250 primes $p \equiv 7 \pmod 8$

Centre Count of $\mathcal{G}_2(\overline{\mathbb{F}}_p)$

Radius: minimal length over all longest directed paths

Centre: collection of vertices for which the furthest distance to any other vertex is at most the radius

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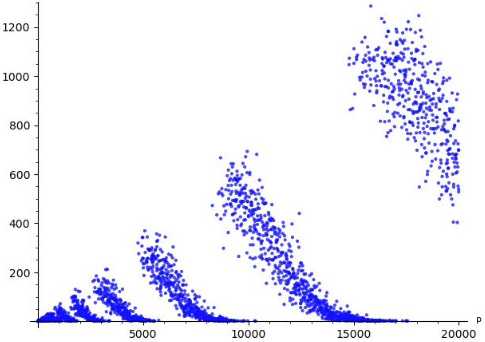
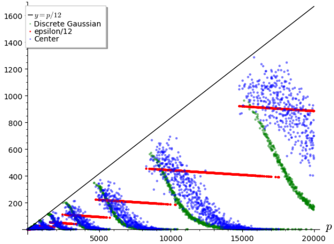


Size of the center of $\mathcal{G}_2(\overline{\mathbb{F}}_p)$ for $5 \leq p \leq 20,000$

Picture for $\ell = 3$ is similar.

Centre Count Explained



Centre Count Explained

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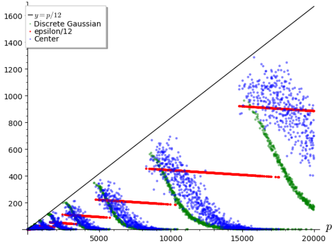
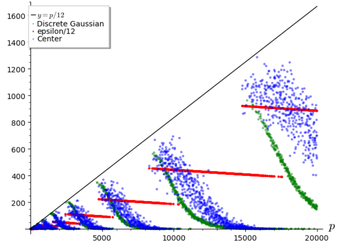


- **Blue**: Centre size of $\mathcal{G}_2(\overline{\mathbb{F}}_p)$

Centre Count Explained



- **Blue**: Centre size of $\mathcal{G}_2(\overline{\mathbb{F}}_p)$
- **Black**: $p/12$ (number of vertices in $\mathcal{G}_2(\overline{\mathbb{F}}_p)$)

Centre Count Explained

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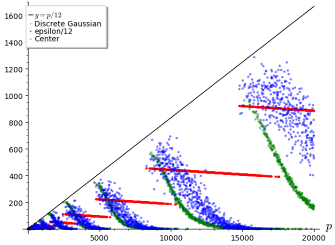

- **Blue**: Centre size of $\mathcal{G}_2(\overline{\mathbb{F}}_p)$
- **Black**: $p/12$ (number of vertices in $\mathcal{G}_2(\overline{\mathbb{F}}_p)$)
- **Green**: discrete Gauß sampling (mean $1.8\log(p)$, standard deviation 0.38) of longest path lengths for a 3-regular graph with $(p-1)/12$ vertices where $p \equiv 1 \pmod{12}$ (thank you, Jonathan Love!)

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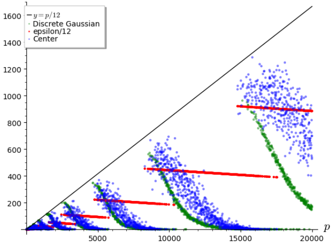

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- **Red**: discrepancy between the theoretically possible and the actual number of ways in which the furthest distance is at most the radius (thank you, Thomas Decru and Jonathan Komada Eriksen!)

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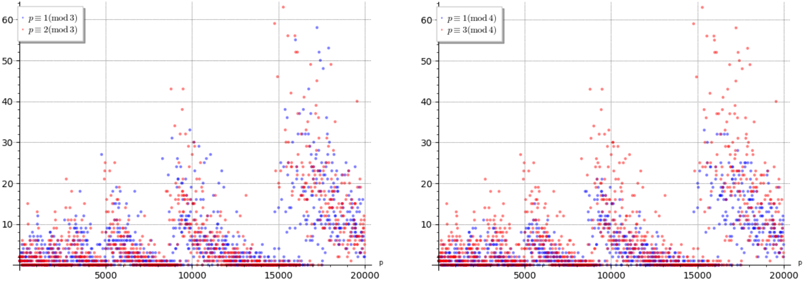

Size of the center of $\mathcal{G}_2(\mathbb{F}_p)$ for $5 \leq p \leq 20,000$

$p \equiv 1 \pmod 3$ | $p \equiv 1 \pmod 4$
$p \equiv 2 \pmod 3$ | $p \equiv 3 \pmod 4$

Observations:

- Centre counts spread out across full range
- Higher centre counts for $p \equiv 3 \pmod 4$ (higher radius values)
- Similar wave pattern as $\mathcal{G}_2(\overline{\mathbb{F}}_p)$ despite Frobenius-conjugate paths

The End!

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Merci! — Questions (ou Résponses)?