When cluster algebras imply (in)finiteness of positive integral points

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Outline

1. Diophantine solutions: a dichotomy
2. Cluster algebras
3. Friezes: Doric orders to point counting
4. Entropy: dichotomy to trichotomy

General picture of the talk

Classification of
Dynkin friezes F

Caldero–Chapoton
Assem–Reutenauer–Smith

Classification of
generalized Cartan matrices C

Killing–Cartan

Finiteness of of
positive integral points $X(\mathbb{N})$
$+$
Positivity of entropy of
arithmetic dynamical systems (X, τ)

$\xleftrightarrow{Z.}$

Classification of
cluster algebras $\mathcal{A}$

$\xleftrightarrow{\text{Fomin–Zelevinsky}}$

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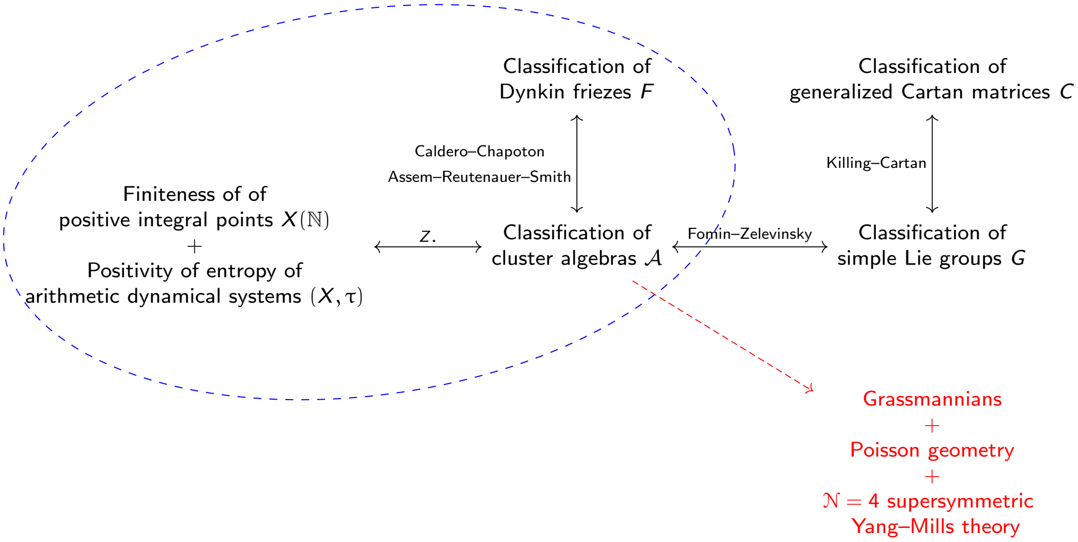
Grassmannians
$+$
Poisson geometry
$+$
$\mathcal{N} = 4$ supersymmetric
Yang–Mills theory

General picture of the talk



The Mordell–Schinzel program over $\mathbb{Z}$

Theorem (Mordell *Acta Math.* 1952)

For any nonconstant $G \in \mathbb{Z}[x, y]$, there are infinitely many integer solutions (x, y, z) to

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Conjecture (Mordell–Schinzel conjecture)

For any $G \in \mathbb{Z}[x, y]$ with $\deg_x G \geq 3$ and $\deg_y G \geq 3$, there are infinitely many integer solutions (x, y, z) to:

$$xyz = G(x, y).$$

The Mordell–Schinzel program over $\mathbb{N}$

Conjecture (Mordell–Schinzel conjecture)

Consider the surface $xyz = G(x, y)$ for $G \in \mathbb{Z}[x, y]$ with $a := \deg_x G$ and $b := \deg_y G$.

- $a, b \geq 3$ $\quad\Rightarrow\quad$ there are infinitely many $\mathbb{Z}$-solutions.

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Theorem (Z. 2025)

Consider the surface $xyz = (x^a + 1)^b + y$.

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Consider the three-fold $xyzw = (x^a + 1)^b y + (x^c + 1)^d z$.

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Theorem (de Saint Germain–Z. in-progress)

Consider the surface $xyz = G(x, y)$ for any $G \in \mathbb{N}[x, y]$ with $\gcd(G, xy) = 1$.

- $ab \geq 4$ and $G(0, 0) = 1$ $\Rightarrow$ there are infinitely many $\mathbb{N}$-solutions.
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General finiteness/infinitude of arithmetic points

The fundamental result in Diophantine geometry about *rational* points:

Theorem (Mordell 1922, Weil 1929, Faltings 1983)

Let X be a smooth projective curve of genus g.

- If $g = 0$, then $\#X(\mathbb{Q}) = 0$ or ∞.
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This talk's main result about *positive integral* points:

Theorem (Z. 2025)

Let X be an n-dim affine cluster variety of type Δ with smallest principal minor t_Δ.

- If $t_\Delta \leq 0$, then $\#X(\mathbb{N}) = \infty$.
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Δ	$\#X(\mathbb{N})$
A_n	$\frac{1}{n+2}\binom{2n+2}{n+1}$
B_n	$\sum_{m=1}^{\lfloor\sqrt{n+1}\rfloor}\binom{2n-m^2+1}{n}$
C_n	$\binom{2n}{n}$
D_n	$\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$
E_6	868
E_7	4400
E_8	26952
F_4	112
G_2	9
∞	∞

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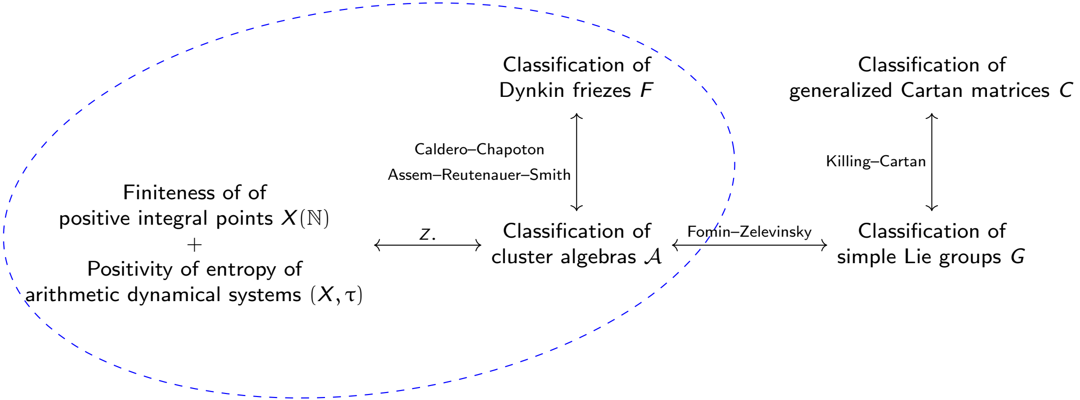
(Sharp except B_n & D_n)

Δ	Height bound
A_n	F_{n+2}
B_n	$2^{\frac{(n+1)^2(n-2)}{2}}$
C_n	F_{2n+1}
D_n	$2^{\frac{n^3}{2}}$
E_6	307
E_7	135503
E_8	2820839
F_4	307
G_2	14
∞	∞

General picture of the talk



Cluster algebras

Simplified notion

A cluster algebra $\mathcal{A} \subset \mathbb{Q}(x_1, \ldots, x_n)$ is a commutative ring

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$$\mathcal{A} = \mathbb{Z}\left[x_1, x_2, x_1', x_2', x_1'', x_2'', x_1''', x_2''', \ldots\right]$$
$$= \mathbb{Z}\left[x_1, x_2, \frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1 x_2}, \frac{x_1+1}{x_2}\right]$$

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This process repeats!

$$\mathcal{A} = \mathbb{Z}\left[x_1, x_2, x_1', x_2', x_1'', x_2'', x_1''', x_2''', \ldots\right] = \mathbb{Z}\left[x_1, x_2, \frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1 x_2}, \frac{x_1+1}{x_2}\right]$$

The cluster algebra $\mathcal{A}$ has 5 cluster variables.

In particular, it is of finite type A_2 with $C = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$.

A modular variety

Let $\mathcal{A}$ be a cluster algebra of type Δ with $n \times n$ generalized Cartan matrix $C = (c_{i,j})$.

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Definition (Lower bound frieze polynomials)

For each $i \in \{1, \ldots, n\}$, consider the polynomial

$$f_{C,i} := x_i y_i - \prod_{j=1}^{i-1} x_j^{-c_{j,i}} - \prod_{j=i+1}^{n} x_j^{-c_{j,i}} \in \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

Define the affine variety $Y_1(\Delta) := \{f_{C,i} = 0\}$ to be the vanishing locus.

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$$Y_1(\Delta)(\mathbb{N}) \longleftrightarrow \{\text{friezes of type } \Delta \text{ over } \mathbb{N}\}.$$

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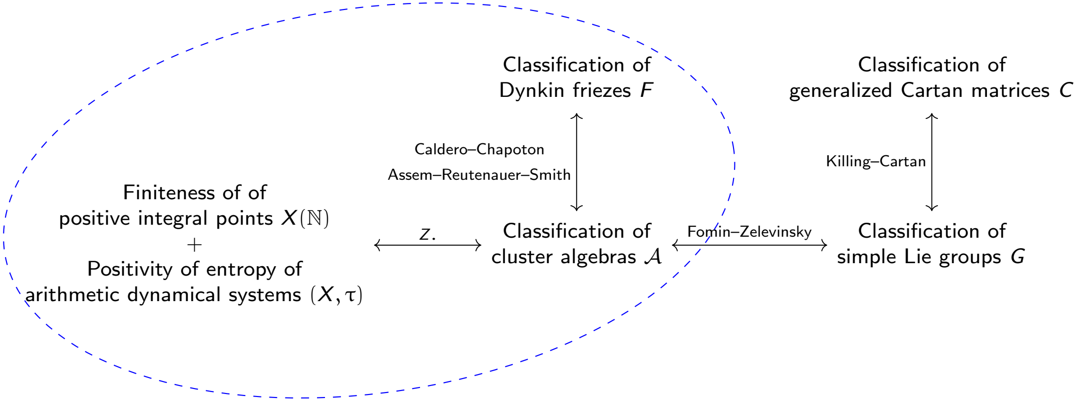
A **frieze** is a $\mathbb{Z}$-algebra homomorphism that sends the generators to positive integers

$$\mathcal{A} \longrightarrow \mathbb{Z}$$

$$\text{cluster variables} \longmapsto \mathbb{N}$$

General picture of the talk



Doric friezes

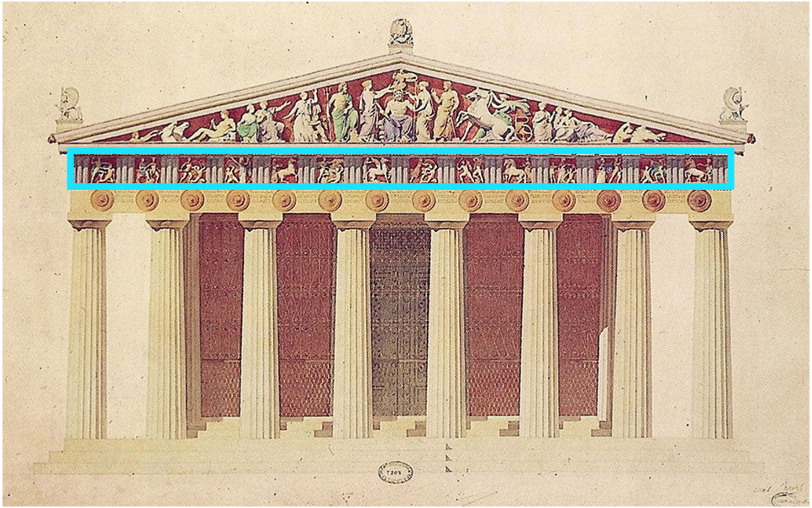


Figure: A Doric frieze (c. 437 BCE) from the Parthenon in Athens

Doric friezes

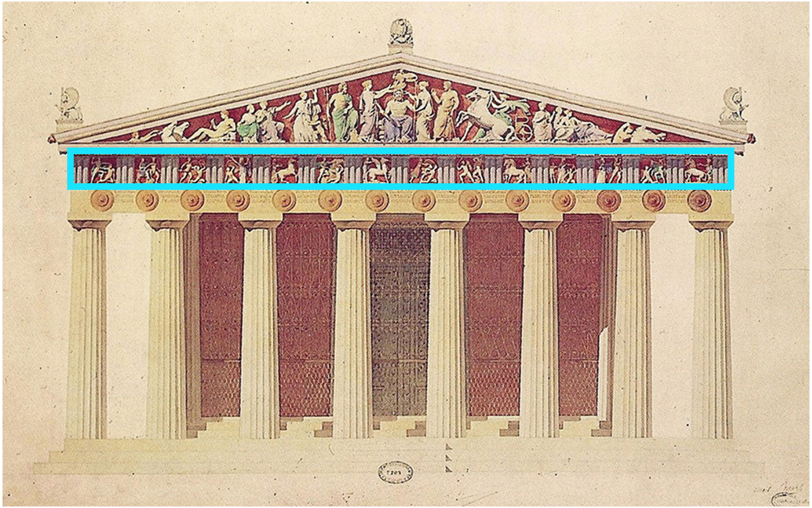


Figure: A Doric frieze (c. 437 BCE) from the Parthenon in Nashville

(Slightly less) classical friezes: *pentagramma mirificum*

A "frieze pattern" is an array of positive integers with an SL_2 unimodular rule.

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Example ($n = 2$ rows)

$$
\begin{array}{ccccccccccccccccc}
\dots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots \\
& \dots & & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & & \dots & \\
\dots & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & \dots \\
& \dots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots &
\end{array}
$$

has all diamonds

$$
\begin{array}{ccc}
& b & \\
a & & d \\
& c &
\end{array}
$$

satisfy the unimodular relation $ad - bc = 1$.

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Theorem (Gauss 1866 posthum.)

All friezes with 2 rows are periodic.

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 & \cdots & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots &
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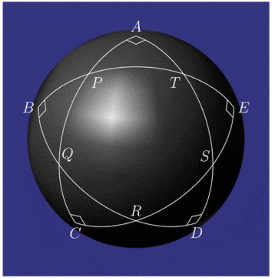

(Slightly less) classical friezes: Coxeter friezes

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A general Coxeter frieze

$$
\begin{array}{ccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & \cdots \\
& \cdots & & x_{1,1} & & x_{1,2} & & x_{1,3} & & \cdots & \\
\cdots & & x_{2,0} & & x_{2,1} & & x_{2,2} & & x_{2,3} & & \cdots \\
& \cdots & & x_{3,0} & & x_{3,1} & & x_{3,2} & & \cdots & \\
\cdots & & x_{4,-1} & & x_{4,0} & & x_{4,1} & & x_{4,2} & & \cdots \\
& \ddots & & \ddots & & \ddots & & \ddots & & \ddots & \\
\cdots & & x_{n,-2} & & x_{n,-1} & & x_{n,0} & & x_{n,1} & & \cdots \\
& \cdots & & 1 & & 1 & & 1 & & \cdots &
\end{array}
$$

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 & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & \\
\cdots & & x_{n,-2} & & x_{n,-1} & & x_{n,0} & & x_{n,1} & & \cdots \\
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& \ddots & & \ddots & & \ddots & & \ddots & & \ddots & \\
\cdots & & x_{n,-2} & & x_{n,-1} & & x_{n,0} & & x_{n,1} & & \cdots \\
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All friezes with n rows are periodic.

Theorem (Conway–Coxeter 1973)

There are $\frac{1}{n+2}\binom{2n+2}{n+1}$ many friezes with n rows.

Dynkin friezes

A “frieze pattern” is an array of positive integers with an SL_2 unimodular rule.

Caldero–Chapoton 2006: equivalently, a homomorphism $\mathcal{A} \to \mathbb{Z}$ for any Dynkin type generators $\mapsto \mathbb{N}$

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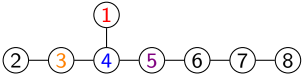
Dynkin diagram for E_8



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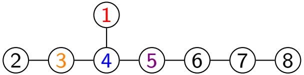

Example of a Dynkin frieze of type E_8

...	1		1		1		1		1		1	...
...		4		3		3		4		4		...
...	15		11		8		11		15		11	...
...	7	41	6	29	5	29	6	41	7	41	6	...
...	16		18		21		18		16		18	...
...		7		13		13		7		7		...
...	3		5		8		5		3		5	...
...		2		3		3		2		2		...
...	1		1		1		1		1		1	...

All diamonds $a \, {}^{b}_{c} \, d$ and $e \, {}^{f}_{\substack{g\\h}} \, i$ satisfy $ad - bc = 1$ and $ei - fgh = 1$.

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...	16		18		21		18		16		18	...
...		7		13		13		7		7		...
...	3		5		8		5		3		5	...
...		2		3		3		2		2		...
...	1		1		1		1		1		1	...

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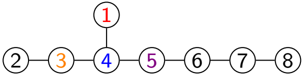

Question: Are there finitely many friezes for each Dynkin diagram?

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...		7		13		13		7		7		...
...	3		5		8		5		3		5	...
...		2		3		3		2		2		...
...	1		1		1		1		1		1	...

All diamonds $\begin{smallmatrix} & b & \\ a & & d \\ & c & \end{smallmatrix}$ and $\begin{smallmatrix} & f & \\ e & g & i \\ & h & \end{smallmatrix}$ satisfy $ad - bc = 1$ and $ei - fgh = 1$.

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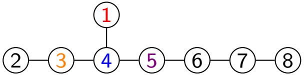

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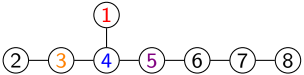

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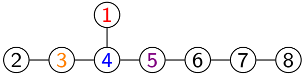

Theorem (Morier-Genoud 2012)

If Δ is an infinite type, then there are infinitely many friezes of type Δ.

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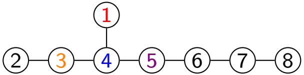

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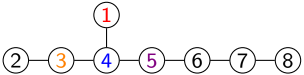

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Theorem (Gunawan–Muller 2022, Muller 2023)

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Enumeration of Dynkin friezes

Question: How many friezes are there for each Dynkin type?

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B_n	$\sum_{m=1}^{\lfloor\sqrt{n+1}\rfloor}\binom{2n-m^2+1}{n}$	Fontaine–Plamondon 2016
C_n	$\binom{2n}{n}$	Fontaine–Plamondon 2016
D_n	$\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$	Morier-Genoud–Ovsienko–Tabachnikov 2012 Fontaine–Plamondon 2016
E_6	868	Cuntz–Plamondon 2021
E_7	?	
E_8	?	
F_4	112	Cuntz–Plamondon 2021
G_2	9	Fontaine–Plamondon 2016
Infinite	∞	Morier-Genoud 2012

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Question: How many friezes are there for each Dynkin type?

Conjecture (Fontaine–Plamondon 2016)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

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Question: How many friezes are there for each Dynkin type?

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Corollary: new proof of frieze counts for any Dynkin type

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Dynkin friezes over integral domains

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Frieze patterns over R
An array $x_{i,j} \in R$ with an SL_2 rule.

Equivalent: a homomorphism $\mathcal{A} \to \mathbb{R}$.

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$$\#\{\text{Friezes of type } A_n \text{ over } \mathbb{F}_q\} = \begin{cases} \frac{q^{n+2}-1}{q^2-1} & \text{if } n \text{ is even,} \\ \frac{(q^{\frac{n+3}{2}}-1)(q^{\frac{n+1}{2}}-1)}{q^2-1} & \text{if } n \text{ is odd and } \mathrm{char}(\mathbb{F}_q) > 2, \\ \frac{(q^{\frac{n+3}{2}}-1)(q^{\frac{n+1}{2}}-1)}{q^2-1} + q^{\frac{n+1}{2}} & \text{if } n \text{ is odd and } \mathrm{char}(\mathbb{F}_q) = 2, \end{cases}$$

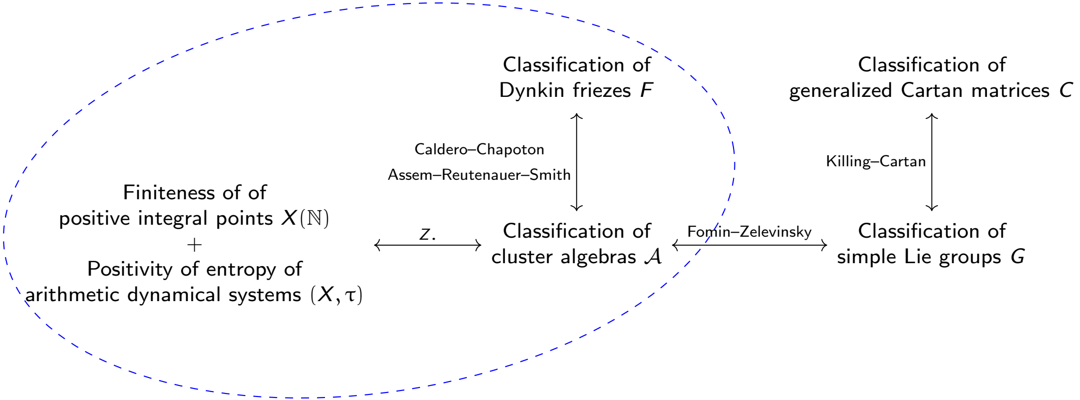
$$\#\{\text{Friezes of type } E_6 \text{ over } \mathbb{F}_q\} = q^6 + q^4 + q^3 + q^2 + 1,$$

$$\#\{\text{Friezes of type } E_8 \text{ over } \mathbb{F}_q\} = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1,$$

$$\#\{\text{Friezes of type } F_4 \text{ over } \mathbb{F}_q\} = q^4 + q^2 + 1.$$

A_n case proved combinatorially by Morier-Genoud 2021 and Short–van Son–Zabolotskii 2025.

General picture of the talk



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The finiteness/infinitude of $X(\mathbb{N})$ realizes the Lie-theoretic *dichotomy*

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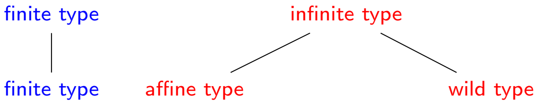

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Theorem (Z. in-progress)

There is a dynamical system (X, τ) on each affine variety of cluster type Δ such that

- if Δ is of finite or affine type, then $h_{\mathrm{alg}}(\tau) = 0$,
- if Δ is of wild type, then $h_{\mathrm{alg}}(\tau) > 0$.

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A finite example

Example (Type G_2)

G_2 has generalized Cartan matrix $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

with $t_\Delta = 1$
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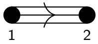

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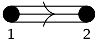


A cluster variety of type G_2 is the surface

$$X := \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{C}^4 \;\middle|\; \begin{aligned} x_1 x_3 &= x_2^3 + 1 \\ x_2 x_4 &= x_1 + 1 \end{aligned} \right\} \subset \mathbb{A}^4_{\mathbb{C}},$$

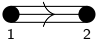
with $\#X(\mathbb{N}) = 9$ and $\#X(\mathbb{Z}) = \infty$.

A finite example

Example (Type G_2)

G_2 has generalized Cartan matrix $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

with $t_\Delta = 1$
and Dynkin diagram



The associated dynamical system

$$X \xrightarrow{\quad \tau \quad} X$$

$$x \longmapsto \left(\frac{x_2^3+1}{x_1}, \frac{x_2^3+x_1+1}{x_1 x_2}, \frac{(x_2^3+x_1+1)^3+x_1^2 x_2^3}{x_1^2 x_2^3 (x_2^3+1)}, x_2 \right)$$

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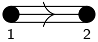

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has entropy $h_{\mathrm{alg}}(\tau) = 0$,
a fixed point, and two 4-cycles in $X(\mathbb{N})$:

$(3,2,3,2)$ $\circlearrowleft \tau$

$(1,1,2,2) \xrightarrow{\tau} (2,3,14,1) \xrightarrow{\tau} (14,5,9,3) \xrightarrow{\tau} (9,2,1,5) \xrightarrow{\tau} (1,1,2,2)$

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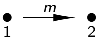
A wild example

Example (A rank-2 wild type)

Let Δ have generalized Cartan matrix $\begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}$

with $t_\Delta = -1$
and Dynkin diagram



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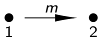

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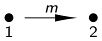

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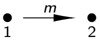

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has entropy $h_{\mathrm{alg}}(\tau) = \log(\frac{3+\sqrt{5}}{2}) \approx 0.96$
and wandering orbits in $X(\mathbb{N})$:

$$\begin{array}{cc} (1,1,2,2) & (1,2,33,1) \\ \downarrow\tau & \downarrow\tau \\ (2,3,122,1) & (33,17,43026,2) \\ \downarrow\tau & \downarrow\tau \\ (122,41,949641,3) & (43026,2531,2,17) \\ \downarrow\tau & \downarrow\tau \\ (949641,23162,\ldots,\ldots) & (2.4\cdot 10^{12}, 9.5\cdot 10^{8},\ldots,\ldots) \\ \downarrow\tau & \downarrow\tau \\ (7.0\cdot 10^{15}, 3.0\cdot 10^{11},\ldots,\ldots) & (3.3\cdot 10^{32}, 3.4\cdot 10^{23},\ldots,\ldots) \\ \downarrow\tau & \downarrow\tau \\ (3.6\cdot 10^{41}, 1.2\cdot 10^{30},\ldots,\ldots) & (1.4\cdot 10^{85}, 4.2\cdot 10^{61},\ldots,\ldots) \\ \downarrow\tau & \downarrow\tau \\ \vdots & \vdots \end{array}$$

The trichotomy in Lie theory, algebra, number theory, and dynamics

Δ	Finite type	Infinite type (affine)	Infinite type (wild)
Generalized Cartan matrix t_Δ	$+$	0	$-$
Lie group	Simple Lie group	Kac–Moody group	Kac–Moody group
Kac–Moody / Lie algebra	f.d.	∞-dim.	∞-dim.
Cluster algebra	f.g.	∞-gen.	∞-gen.
Diophantine solutions $X(\mathbb{N})$	Finite	∞	∞
Algebraic entropy of τ	0	0	$+$